Pricing on Markets with Network Externalities and Coalitionally Rational Consumers.

Attila Ambrus* and Rosa Argenziano†‡

Preliminary Draft.
Do not quote or distribute without contacting the authors.

August 31, 2003

Abstract

This paper analyzes pricing decisions in both monopoly and duopoly settings on markets that exhibit network externalities. A two-stage game is presented where first firms announce prices, then consumers simultaneously decide whether to join a network or stay out. We characterize subgame perfect equilibria which satisfy the condition that consumers play coalitionally rationalizable strategies in every subgame. This corresponds to assuming that consumers can successfully coordinate their actions whenever it is in their joint interest to do so, as long as it does not require explicit coordination. On one-sided markets and on two-sided markets with homogeneous consumers the assumption guarantees that the monopolist attains the maximum profit compatible with equilibrium. In Bertrand competition it drives participants' profits to zero, re-establishing the classic result on markets without externalities. On one-sided markets the equilibrium price has to be zero, but on two-sided markets one side can be subsidized while the other pays positive price. If consumers are heterogeneous with respect to how much they care about the network effect, on two-sided markets multiple active networks can arise that are priced differently, despite the assumption of coalitional rationality. In particular, the monopolist might provide two separate networks to price discriminate consumers more efficiently. One of them is relatively cheap on one side of the market while the other is on the opposite side. The cheap network sides

*Harvard University; e-mail: ambrus@fas.harvard.edu

†Yale University; e-mail: rosa.argenziano@yale.edu

‡We would like to thank Dirk Bergemann, Erica Field, Dino Gerardi and Stephen Morris for useful comments.

attract a large number of low type consumers, which makes high types willing to go to the expensive sides. This way product differentiation is endogenized in equilibrium.

1 Introduction.

A market has network externalities if consumers' utility from purchasing a product on the market depends on which other consumers buy the same product. The simplest example of this is when a consumer's utility is increasing in the total number of consumers buying the same product. The standard terminology for this case in the literature is one-sided markets with network externalities, and it provides a reasonable framework to model markets like telecommunication services (phone, mobile phone). Another highlighted case is when consumers are divided into two distinct subgroups, and a consumers's utility on one side is increasing in the total number of consumers on the other side of the market who buy the same product. These are labeled two-sided markets with network externalities and can be used to model situations in which two groups of agents need a common platform to interact and one or more firms own platforms and sell access to them. The higher the number of agents on one side who join a platform, the higher the utility of an agent on the other side who joins the same platform because he has a higher number of potential partners to trade or interact with.

One example is the market for on-line matchmaking services where the two sides are women and men. In other cases, the two sides are buyers and sellers, and the platforms are auction websites, yellow pages services, classified ads, credit card networks (the sellers are merchants who accept the credit card and the buyers are credit card holders) and video game and computer platforms (the sellers are software developers for the platform and the buyers are end users of the platforms). A labor market example is temporary work (TEMP) agencies where the two sides are employers and employees.

This paper investigates the pricing decisions of firms operating on markets with network externalities. We refer to these firms as network providers. Equilibrium analysis on markets with networks is an involved task. The coordination problems that arise among consumers result in a severe multiplicity of equilibria in both games in which there is a monopolist network provider and in price competition games among multiple providers. Consumers can have various self-fulfilling expectations on which networks other consumers join and whether they join some network at all.

Coordination failures and the resulting inefficiencies are relevant phenomena on markets with network effects. After all there are typically many anonymous consumers on these markets who cannot communicate to each other and explicitly coordinate before joining a network. Nevertheless, in some cases it is reasonable to expect consumers to be able to coordinate on one network versus the other. The simplest example is that if a network is cheaper for all potential consumers than another one and consumers do not have inherent preferences favoring one network versus the other, then choosing the cheaper network becomes a natural candidate for consumers to coordinate on.

The central assumption of our paper is that consumers are able to coordinate their decisions on the market to their advantage if they interests coincide and if coordination can be achieved without communication, like in the case of a cheap and an expensive network. But in cases in which explicit coordination would be needed we do not assume that coordination is achieved, even if it is in the common interest of consumers. An example is when there are two networks charging the same price and it is in the best interest of consumers to all join the same one, but it is unclear which one should be that network.

The formal concept we use to incorporate this assumption in the analysis is coalitional rationalizability, proposed by Ambrus (2002). The main idea behind the concept is that players can make implicit coalitional agreements among each other to confine play to a subset of strategies if it is in their mutual interest to do so. An agreement is in the mutual interest of participants if for each of them the following is true. Even the most pessimistic expected payoff she can expect if the agreement is made is strictly higher than the most optimistic expected payoff she can hope for if the agreement is not made and in particular if he chooses an action that is incompatible with the agreement. Agreements can be both of the form of agreeing upon not doing a certain action (like not staying out of the market) and specifying a certain action for each participant (like specifying a network to join). The concept also allows for a series of agreements in an iterative manner, since once an agreement is made by a group of consumers, another agreement might become unambiguously in the interest of another group. This again uses the fact that these agreements are based on public information (the payoffs of the game) and therefore every consumer can infer what agreements are to be made, without communicating to each other.

Our analysis is done in an extensive form game in which first firms (or just one firm in case of a monopoly) announce prices they charge for using their networks. Firms can charge different prices on different sides of the market. Then consumers observe these prices, and finally all consumers simultaneously and independently choose networks or decide to stay out of the market. We characterize the subgame perfect equilibria of this game that are compatible with coalitional rationalizability.

The first set of results establish that the assumption of coalitional rationality makes some classic results of pricing games true for markets with network externalities if either the market is one-sided or if it is two sided but consumers are homogeneous on both sides. The monopolist gets the highest possible profit compatible with subgame perfect equilibrium, and therefore can use coalitional rationality of consumers in his advantage. To sharp contrast, in Bertrand type competition all coalitionally rationalizable subgame perfect equilibria yield zero profit to both firms. Furthermore, it is guaranteed that no consumers with positive evaluation stay out of the market. On one-sided markets price competition brings equilibrium prices down to zero. On two-sided markets, however, one side of the market can be subsidized while the other pays a strictly positive price.

The second set of results show that introducing heterogeneity with respect to how much consumers care about the network effect introduces new phenomena to price setting games. Consumers on the same side can be price discriminated despite the networks are not differentiated, the types of consumers are not observable by firms and the retained assumption of coalitional rationalizability. The latter is particularly surprising since the type of heterogeneity is such that all consumers on the same side of the market still have the same interests - the utility for all of them is higher the lower is the price to be paid and the higher is the number of consumers from the other side joining the network. Only the relative weight of these two can differ among consumers. We consider a simple form of heterogeneity in which both sides have two type of consumers and one type cares more about the network externality (that is how many people join the network from the other side) than the other.

We show that the monopolist can either target all consumers, only high consumers, or all consumers on one side and only high consumers on the other side, depending on the fraction of high versus low types and on how much high and low types care about the external effect. Furthermore, if there are relatively few high types and their evaluation is very high compared to low types, it can be optimal for the monopolist to establish two networks that are physically the same, but priced differently. For this to be compatible with coalitional rationalizability it has to be the case that a network that is cheaper on one side of the market has to be more expensive on the other side.[1] In these equilibria the high types of one side are together on the same

[1] In particular, the monopolist cannot create a high price network for only high type consumers and a cheaper network for low types. Coalitional rationality in that case would imply that high types would switch over to the cheap network as well.

network with the low types of the other side and vice versa. The network that is cheaper on one side has more consumers joining it in equilibrium.

The way price discrimination is achieved in these equilibria is through endogeneous product differentiation. Networks are physically the same, but low types are only willing to join the networks that are relatively cheap, and since there are a lot of them, it makes the network more valuable for consumers on the other side. Henceforth high types on the other side are willing to choose the network despite it is more expensive for them.

We are currently working on analyzing price competition on two-sided markets with network externalities and heterogeneous consumers.

The structure of the paper is the following. Section 2 describes the related literature. Section 3 briefly introduces coalitional rationalizability. Section 4 investigates both price setting by the monopolist and Bertrand competition between two firms on one-sided markets with network externalities. Section 5 provides similar analysis for two-sided markets with homogeneous consumers on each side. Section 6 introduces heterogeneity among consumers in two-sided markets and characterizes the equilibria of the monopolist price setting game. Finally, section 7 concludes and describes future research directions.

2 Related Literature.

The first analysis of consumer demand in the presence of network externalities appears in Leibenstein (1950) and the related problem of multiple market sizes compatible with the same price was first discussed by Rohlfs (1974) in the context of a monopolistic market for communications services. The seminal works of Katz-Shapiro (1985) and Farrell-Saloner (1985) inspired a large literature that addressed multiple issues associated with markets with one-sided network externalities[2]. The first formal analysis of oligopolistic competition on these markets, using a Cournot framework, appears in Katz and Shapiro (1985). De Palma-Leruth (1993) and Bental-Spiegel (1995) investigate Bertrand competition in the presence of network externalities. De Palma-Leruth (1993) assumes that consumers are homogeneous and that firms can horizontally differentiate their products. We allow for heterogeneity in how much consumers care about the network effect, but not for product differentiation. Bental-Spiegel (1995) introduces heterogeneity among

[2]For a wider survey on this literature, see Farrell-Klemperer (2001), Katz-Shapiro (1994) and Economides (1996)

consumers, but only with respect to their income.

Recently, a number of papers investigated the issue of optimal pricing and price competition on markets with two-sided network externalities. Rochet-Tirole (2002a) and Schmalensee (2001) analyze credit card systems with a monopolist network while Laffont et al (2001) analyze competition between internet backbone operators[3]: in this case, since all the backbones are connected, there is no specific benefit for those agents who join the largest backbone. Parker-Van Alstyne (2000) analyze "hardware" markets where the two groups of agents who purchase a firm's products are content producers who buy the tools to produce contents compatible with a given hardware, and end-users who buy the hardware. They describe networks' pricing strategies and show how they are affected by the intensity of the network externalities across groups, up to the point that firms might optimally subsidize one of the two sides. Ellison et al (2002) study competition between two auction sites and show how the relative size of the two sides of this market (sellers and buyers) affects the range of equilibrium market shares.

The papers most closely connected to our analysis are Caillaud and Jullien (2001a) and (2001b), Rochet and Tirole (2002b) and Jullien (2001). Caillaud and Jullien (2001a) and (2001b) analyze markets where firms are intermediaries offering matchmaking services to two groups of agents. They assume that consumers on each side are homogeneous and analyze both optimal pricing by a monopolist and price competition between two intermediaries. They reduce the issue of multiple equilibria by imposing monotonicity on the demand function of consumers. Coalitional rationality on the other hand is an assumption directly on the expectations of consumers.

Rochet and Tirole (2002b) also study monopolistic pricing and price competition between two firms on markets where the firms are platforms who try to attract two groups of agents. They assume that agents on each side are heterogeneous with respect to the gross surplus they derive from the interaction with the other group and that networks' primary pricing instrument is a transaction fee they charge when two agents interact using the platform. Instead, we assume that firms charge a registration fee to agents who choose to access their networks. The assumption that networks use (only) a transaction fee eliminates the issue of multiple self-fulfilling expectations in the monopoly case because even if the gross surplus of consumers of one side who join the network is increasing in the number of agents of the other side who join as well, their aggregate demand is independent from the

[3] The two sides on the market are websites and consumers.

network size. As for the duopoly case, they derive the optimal pricing structure adopted by the networks focusing only on symmetric equilibria where both firms charge the same prices, under different assumptions on the governance structure. Finally, Jullien (2001) constructs a duopoly model that allows for more than two subgroups of consumers and for both inter-groups and intra-group network externalities. This paper also differs from ours in that it assumes that the intrinsic value of the good sold by each firm is high compared to the network effect and also that one of the firms is highlighted in the sense that consumers always coordinate on the equilibrium which is the most favorable for this firm.

Our model allows for multiple consumer groups on each side of the market and for multiple networks with different prices (different types of consumers might select to join different networks). In this aspect, our analysis is connected to the literature on price discrimination (for an overview see Varian (1987)) multiproduct pricing (see Baumol et al. (1982)) and the theory of screening (for an overview see Salanie (1997)).

3 Coalitional Rationality.

The central assumption of our paper is that consumers can coordinate their actions whenever it is unambiguously in their interest and it does not require communication. Formally, we require consumers to play coalitionally rationalizable strategies, as defined in Ambrus (2002), in every subgame that follows the price announcements by firms.

Coalitional rationalizability is a reasoning procedure that iteratively eliminates certain strategies from the strategy space. Steps of this elimination procedure correspond to agreements by groups of players (coalitions) to confine play to a certain subset of their strategies. It is analogous to iterative removal of never best response strategies, but in this procedure groups of players, not only individual players, can jointly eliminate strategies. The set of coalitionally rationalizable strategies are those which survive this procedure.

In each round of the procedure every coalition makes every agreement which is unambiguously in the mutual interest of all coalition members. These agreements are called supported restrictions and they are defined formally below. Readers not interested in the technical details of how coalitional rationalizability is defined are advised to skip the paragraph and read the intuitive summary of the definition following the formal treatment.

Let I be the set of players in the game, $S = \underset{i \in I}{\times} S_0$ the set of strategies and $u_i\ \forall\ i \in I$ the payoff functions. For every $i \in I$ let Ω_{-i} be the set of probability distributions over S_{-i}, representing possible conjectures of player i on strategies of the other players. For an arbitrary product subset A of the strategy space let $\Omega_{-i}(A)$ be the set of probability distributions f_{-i} for which $f_{-i}(s) > 0$ only if $s \in A$. Let $u_i(s_i, f_{-i})$ denote the expected payoff of player i if he plays strategy s_i and his conjecture on other players' strategies is f_{-i}. For every $f_{-i} \in \Omega_{-i}$ let $BR_i(f_{-i})$ denote the set of best responses of player i to conjecture f_{-i}. For every $J \subset I$, $i \in J$ and $f_{-i} \in \Omega_{-i}$ let f_{-i}^J denote the marginal distribution of f_{-i} over S_J. Finally, let $b_i(f_{-i}) = u_i(s_i, f_{-i})$ where $s_i \in BR_i(f_{-i})$ (any best response strategy gives the same expected payoff, so b_i is well-defined.

Let A and B be two nonempty product subsets of the strategy space such that $B \subset A$. Let $J \subset I$.

Definition: B is a *supported restriction* by J given A if

1) $B_i = A_i$, $\forall\ i \notin J$, and

2) $\forall\ j \in J$, and $\forall\ f_{-j}$ for which $\exists\ s_j \in \Omega_{-j}(A_j/B_j)$ such that $s_j \in BR_j(f_{-j})$ it is the case that

$b_j(f_{-j}) < b_j(g_{-j})\ \forall\ g_{-j}$ such that $g_{-j} \in \Omega_{-j}(B)$ and $g_{-i}^{I/J} = f_{-i}^{I/J}$.

The first condition in the definition of supported restriction requires that only the strategies of those players who are members of the given group are restricted. The second condition requires that for any player in the coalition, any belief against which he has a best response strategy outside the agreement yields a strictly lower expected payoff than any belief that is consistent with other players in the coalition keeping the agreement, holding the marginal expectation concerning the strategies of players outside the coalition fixed. To state it simply, for a restriction to be supported, we require that if the restriction is not made, every player in the coalition either plays inside B, or wishes that the restriction was made, because he expects a lower payoff than any payoff he could expect if the agreement was made.

The set of coalitionally rationalizable strategies is then defined as follows. Let $z(A)$ be the set of all supported restrictions by any coalition given A. Let $A^0 = S$ and define A^k for $k = 1, 2, ...$ iteratively as follows: $A^k = \bigcap_{B \in z(A^{k-1})} B$. The set if coalitionally rationalizable strategies is defined to be $\bigcap_{k=0,1,...} A^k$.

Players start from the assumption that any strategy is possible to be played. At every step of the procedure every supported restriction by every coalition is made. Therefore the next step starts out from the intersection of all these supported restrictions. Finally, the strategies that survive the iterative procedure are called coalitionally rationalizable.

For a more detailed description of coalitional rationalizability see Ambrus (2002).

In this paper we analyze extensive form games in which one set of the players move first and then another set of players choose actions after observing the previous moves. The first set of players are firms setting prices and the second set of players are the consumers who decide which firm's network to join or whether to stay out of the market. We make the assumption that consumers only play coalitionally rationalizable strategies in every subgame of this extensive form game and focus on subgame perfect Nash equilibria of these games that are compatible with the above assumption. Therefore we combine coalitional rationalizability with equilibrium analysis and use it to select reasonable equilibria in these market games. Our interpretation is that consumers make implicit agreements that are in their interest and do not require explicit coordination. Then they simplify the game by restricting the set of strategies to those which are compatible with these agreements. We investigate the subgame perfect Nash equilibria of the game which is obtained from the original extensive form game by simplifying every subgame the way described above.

We call the resulting solution concept coalitionally rationalizable subgame perfect equilibrium (crspe).

Coalitional rationalizability does not put extra restrictions on the actions of firms in these games because their interests are in conflict with each other. Therefore the results would be exactly the same if we used the concept of extensive form coalitional rationalizability proposed by Ambrus (2003) and which requires all players in the game to be sequentially coalitionally rational. We chose the current form of presentation for ease of exposition.

To provide some intuition on how the concept works, we provide some examples of supported restrictions in the games that we analyze.

Suppose there are two firms on the market, A and B, operating one network each. In each example every consumer who does not join any network gets zero utility.

In the first example assume that the market is one-sided and there is a continuum of consumers all getting utility $x - p$ if joining a network that charges price p and which a total number x of consumers join. Assume both firms charge 0 price. Then the agreement to join either A's network or B's network (or agreeing upon not to stay out of the market) is a supported restriction for the coalition of all consumers, because if the agreement is made, then any possible conjecture that is compatible with the agreement is such that a best response to it yields expected payoff of at least 1/2 (the conjecture should allocate an expected size of at least 1/2 to one of the two networks), while staying out of the market yields zero payoff. Because prices charged by the two networks are the same, no more strategies are eliminated by coalitional rationalizability in this subgame. Both joining A's network and joining B's network are coalitionally rationalizable for every consumer and therefore this subgame has three coalitionally rationalizable Nash equilibria. Either every consumer joins A or every consumer joins B, or one half of them join each network.

Consider now the same market, but assume A charged a price of 1/4, while B charged a price of 1/2. Then restricting play to joining A is a supported restriction for the coalition of all players, since it yields payoff 3/4 to all consumers, while joining A's network can yield a payoff of at most 1/2 and staying out yields zero. Coalitional rationalizability pins down a unique strategy profile in this subgame.

We conclude by noting that the solution concept that we use is not equivalent to coalition-proof Nash equilibrium (Bernheim, Peleg and Whinston (1987)) in the games that we analyze. In particular, consider the subgame in our first example above. As shown, the profile in which half of the consumers join A's network and the other half join B's is a coalitionally rationalizable Nash equilibrium of the game. It is not a coalition proof Nash equilibrium though, since, for example, the players who join A in this proposed profile could jointly deviate to B and be all better off, without any subgroup of them wanting to deviate further. Our point though is that players who join B in the proposed profile have the same incentives to switch and without explicit coordination it is not obvious whether a player should switch or stay. Coalition-proof Nash equilibrium takes the position that players can successfully coordinate even in those subgames where explicit coordination is needed, while we only require it in cases in which explicit coordination is not necessary, keeping in mind that most applications of our model are such that there is a large number of consumers who do not regularly interact with each other.

4 One-sided Markets.

Although the main focus of the paper is monopolistic pricing and oligopolistic price competition on two-sided markets with network externalities, first we provide the same analysis for one-sided markets with network externalities. We do it partly for the sake of completeness and partly for expositional concerns. Coalitional agreements among consumers are simpler in this context and therefore it is easier to grasp the intuition behind our results.

Following the convention, by one sided markets with network externalities we mean a market where each consumer's utility from using a network is increasing in the total number of consumers using the network. We show that while a wide variety of subgame perfect Nash-equilibria exist both in the case of a monopolist and in the case of Bertrand competition between two firms, only few of them survive the additional requirement of being stable with respect to coalitional agreements.

Let there be a continuum of consumers with mass 1. Assume that a consumer gets utility 0 if she does not join any network, while she gets utility $xu - p$ if she joins a network that charges a registration fee p and which a total number of x consumers join. Note that this specification implies that consumers do not have any inherent preference for one network versus the other. The parameter u is individual-specific and we refer to it as intensity parameter. Assume that it is distributed among individuals according to distribution function F with bounded support. Let $M > 0$ be an upper bound of the support. Assume $F(u) = 0$ for every $u < 0$ (no consumer gets negative utility from using the network) and that there is $e > 0$ such that $F(e) < 1$ (there are consumers who can get positive utility from using the network). A highlighted special case is when there is some $\mathbf{b} > 0$ such that $F(u) = 0$ for every $u < \mathbf{b}$ and $F(u) = 1$ for every $u > \mathbf{b}$. This corresponds to the case when consumers are homogeneous and their intensity parameter is $\mathbf{b}$.

Finally, assume that the marginal cost of serving an additional consumer is zero.

4.1 Monopoly.

Consider first the case where there is one firm operating one network on the market. We analyze the game in which first the firm sets registration fee p, then consumers observe this and finally they simultaneously decide whether to join the network or not. Formally, the firm's strategy set is $\mathbb{R}$, while every consumer's strategy is a function that allocates either 0 (not joining) or 1 (joining) to every $p \in \mathbb{R}$.

There are many subgame perfect Nash equilibria of this game, depending on the expectations of consumers after different price announcements.

For example, if consumers are homogeneous with intensity parameter

$\hat{v} > 0$ then every $p \in [0, \hat{v}]$ can be equilibrium price and for any such p there are equilibria in which a mass of $\frac{p}{\hat{v}}$, 0 or 1 people join the network. The sketch of the proof is that if after any nonnegative price announcement besides p no consumer joins the network (which is compatible with subgame perfect Nash equilibrium since if no other consumer joins then it is optimal for a consumer not to join), then it is optimal for the firm to set the price to be p. And then $\frac{p}{\hat{v}}$, 0 or 1 people joining the network are all compatible with consumers being in optimum.

For any distribution function F it holds that there are subgame perfect Nash equilibria in which the firm gets zero profit. This happens if for any positive price announcement no consumer joins the network, in which case it is optimal for the firm to set $p = 0$ (or set a positive price, with no consumer joining the network).

Instead, the assumption that consumers are coalitionally rational guarantees that the monopolist gets the maximum profit possible in subgame perfect equilibrium and if our assumptions hold then this profit is strictly positive.

Let crspe stand for coalitionally rational subgame perfect Nash equilibrium.

Theorem 1 *Let s be any subgame perfect equilibrium profile of the game with one firm and let π be the profit that the firm gets in this equilibrium. Let s' be any crspe profile of the game with one firm and let π' be the profit that the firm gets in this equilibrium. Then $\pi' \geq \pi$ and $\pi' > 0$.*

Proof: see appendix.

In the case of homogeneous consumers, there is a unique equilibrium for the game with one firm, in which the firm sets the price equal to the intensity parameter of consumers and every consumer joins the network. The firm extracts the maximum possible gross consumer surplus.

Theorem 2 *Suppose $\exists\ \hat{v} > 0$ such that $F(u) = 0$ for every $u < \hat{v}$ and $F(u) = 1$ for every $u \geq \hat{v}$. Then crspe is unique up to a set of consumers with 0 measure. In the crspe the firm sets a price equal to $\hat{v}$ and every consumer joins the network if the price is less than or equal to $\hat{v}$ and no consumer joins the network if the price is larger than $\hat{v}$.*

Proof: see appendix.

4.2 Duopoly.

Suppose now that there are two firms operating networks, A and B. The game has two stages. In the first, firms simultaneously and non-cooperatively announce prices. In the second each consumer observes prices and decides whether to join network A, join network B, or stay out.

Again there is a severe multiplicity of subgame perfect Nash equilibria, due to the fact that for any couple of prices announced by the firms there exist many sets of self-fulfilling expectations. In some of them no consumer joins any network, in others only one firm's network attracts consumers and in yet others some consumers join A while others join B. In the latter case the networks can charge the same price and have the same size, but it can also be that they charge different sizes and have different sizes. The prices that firms charge can be strictly positive and equilibrium profits can be strictly positive for one or both firms. Any fraction of consumers can enter the market, from zero to one. For a complete characterization of subgame perfect Nash equilibria in the case of homogeneous consumers, see appendix.

The next theorem establishes that the assumption that consumers are coalitionally rational, and therefore coordinate their actions whenever it is unambiguously in their interest, sharply reduces the set of equilibria. In all crspe of the price competition game both firms get zero profit, therefore coalitional rationalizability reestablishes the classic Bertrand result for one-sided markets with network externalities.

Let x denote the number of consumers joining network A and y the number of consumers joining B in equilibrium, p the price charged by A and q the price charged by B.

Theorem 3 *In every coalitionally rational subgame perfect equilibrium of the price competition game, both firms charge* 0 *price. Moreover, there are two types of such equilibria:*

1. *all consumers with positive* u_i *parameter choose firm* A *and* $x \geqslant y$ *(or, analogously, all consumers with positive* u_i *parameter choose firm* B *and* $x \leqslant y$*) ;*

2. *all consumers with positive* u_i *parameter join some network and* $x = y$*. In unreached subgames* $x = 0$ *if* $p > q$ *and* $y = 0$ *if* $q > p$*.*

Proof: see appendix.

Note that in every crspe all consumers with positive intensity parameter join some network.

In the special case of homogeneous consumers the above theorem implies that both firms set a zero price, that every consumer joins some network, and either all of them join A, or all of them join B, or exactly half of them join each firm. The intuition behind the above result is that coalitional rationality puts a restriction on beliefs when the prices of the two networks are different. Consider a case when homogeneous consumers observe prices $p < q$ where $0 < p$ and $q < u$. Subgame perfect Nash equilibrium can be compatible with consumers choosing either networks or both, or staying out of the market. But if all consumers implicitly agree to join A, they are all strictly better off than if they joined B or stayed out. Therefore joining the cheaper network is a supported restriction for the consumers and coalitional rationalizability implies that the more expensive network will be empty. Through this mechanism, undercutting strategies become very effective and essentially this is what re-establishes the Bertrand result. Furthermore, if both prices are zero, then excluding the strategy "staying out" is a supported restriction for the coalition of consumers with positive intensity parameter, because it guarantees a positive expected payoff for them, while staying out gives a payoff of zero. This is why in every crspe all consumers with positive evaluation join some network.

5 Two-sided Markets with Homogeneous Consumers.

This section investigates optimal pricing on two-sided markets with network externalities, that is markets where consumers are divided into two distinct groups and the utility of using a network for a consumer belonging to one group is increasing in the number of consumers from the other group using the same network.

Assume there is a continuum of side 1 consumers and a continuum of side 2 consumers, both of mass one. Let the marginal cost of serving an additional consumer be zero for a network provider. We also assume that firms can distinguish between consumers on the two sides and therefore can charge a different registration fee to side 1 and side 2 consumers. The utility of a side 1 consumer by joining a network which charges price p_1 to side 1 consumers and which has x_2 side 2 consumers joining it is $ux_2 - p_1$ where $u > 0$. Similarly, the utility of a side 2 consumer by joining a network which charges price p_2 on side 2 consumers and which has x_1 side 1 consumers joining it is $ux_1 - p_2$. The utility of a consumer not joining any network is 0.

5.1 Monopoly.

First, we consider the case when there is one firm providing network service on the market.

We model the market by the following game. First, the monopolist sets a pair of prices, one for side 1 consumers and one for side 2 consumers. Then, consumers observe these prices and simultaneously decide whether to join the network or not. Therefore the set of strategies of the firm is R^2 while for each consumer the set of strategies is $\mathsf{R}^2 \to \{0, 1\}$.

The first theorem characterizes the set of subgame perfect Nash equilibria of this game. As opposed to the case of one-sided markets with network externalities the monopolist is guaranteed to get a strictly positive profit in subgame perfect Nash equilibrium. The intuition is that the firm now has the option of getting around the coordination problem by charging a slightly negative price on one side of the market and thereby guaranteeing that all consumers of that side join the network. But then he can charge a price close to u on the other side of the market and still have all consumers joining the network. This way the monopolist can achieve a profit arbitrarily close to $u > 0$. The theorem below shows that the monopolist's profit can be anything between u and $2u$. This implies that in every subgame perfect Nash equilibrium some consumers join the network. But it is not true that in all subgame perfect Nash equilibrium all consumers join the network.

Theorem 4 *The subgame perfect nash-equilibria of the monopolistic model of a two-sided market with symmetric, homogeneous network externalities are the following:*

"Monopoly equilibria with full coverage" *characterized by market shares* $x_1 = x_2 = 1$ *and prices* $p_1, p_2 \in [0, u]$, $p_1 + p_2 > u$

"Monopoly equilibria with partial coverage" *characterized by market shares* $0 < x_1 + x_2 < 2$ *and prices* $p_1, p_2 \in (0, u)$ *such that* $x_i = \frac{p_{-i}}{u}$ *for* $i = 1, 2$ *and* $p_1 x_1 + p_2 x_2 > u$

The profit of the firm in subgame perfect Nash equilibrium can be any $\pi \in [u, 2u]$.

Proof: see appendix.

Despite the fact that the monopolist is guaranteed to get a profit of at least u, there is continuum of subgame perfect Nash equilibria and the monopolist's profit depends on the expectations of consumers. The next theorem shows that among these equilibria the ones satisfying coalitional rationality yield the maximum possible profit that is compatible with subgame perfect Nash equilibrium. All these equilibria are outcome equivalent and along the equilibrium path the monopolist charges u on both sides and all consumers join the network. Therefore the profit of the monopolist is $2u$. These results are analogous to the ones in the previous section.

Theorem 5 *In every crspe of the game with one firm and homogeneous consumers the monopolist charges u on both sides of the market and all consumers join the network.*

Proof: see appendix.

We note that if consumers are coalitionally rational then the monopolist cannot increase its profit by establishing two networks. It already extracts the maximum possible consumer surplus by operating one network and charging u on both sides of the market. In fact, if the firm operates two networks and both are active (a positive fraction of consumers join each) then its profit must be strictly less than $2u$, since gross consumer surplus is then less than $2u$ and in equilibrium no consumer can have negative utility.

5.2 Duopoly.

Assume now that there are two firms on the market. Consider the game in which first firms simultaneously announce prices for both side 1 and side 2 consumers, then consumers observe these announcements and finally consumers on both sides of the market simultaneously decide whether to join a network and if yes, which one.

Let p_k, $k = 1, 2$ denote the price that A charges on consumers of side k and q_k, $k = 1, 2$ denote the price that B charges on consumers of side k.

Theorem 6 below characterizes the set of subgame prefect Nash equilibria of this game. The set of equilibria of the price competition game is somewhat less rich than in the one-sided market case. In particular in all subgame perfect Nash equilibrium firms get the same profit. If both firms are active, then in equilibrium they charge the same prices. Finally there is either zero or full consumer participation. The intuition behind these results is that certain deviations that get around the coordination problem consumers face might be profitable in this setting, restricting the type of subgame perfect equilibria of the model. In particular, if a firm announces a price for one side of the market that makes choosing its network a dominant choice for consumers on that side (in particular the price is negative and less than the competing network's price by more than u) then he can charge a strictly positive price on consumers on the other side and still make sure that all consumers on that side join its network too[4].

Let x_k $k = 1, 2$ denote the number of consumers joining network A and y_k $k = 1, 2$ denote the number of consumers joining network B in a given equilibrium.

Theorem 6 *The set of spne of the game of price competition between two firms on a two-sided market with symmetric, homogeneous network externalities is the following:*

"Pessimistic equilibria" *characterized by* $x_1 = x_2 = y_1 = y_2 = 0$ *and* $p_1 = p_2 = q_1 = q_2 = 0$;

"Symmetric equilibria with full coverage and positive prices" *characterized by* $x_1 = x_2 = y_1 = y_2 = \frac{1}{2}$ *and* $p_1 = p_2 = q_1 = q_2 = \hat{p} \in \left[0, \frac{u}{2}\right]$;

"Symmetric equilibria with full coverage and cross-subsidies" *characterized by* $x_1 = x_2 = y_1 = y_2 = \frac{1}{2}$ *and* $p_i = q_i = \hat{p} \in \left[0, \frac{u}{2}\right]$, $p_{-i} = q_{-i} = -\hat{p}$ *for i=1,2;*

"Monopoly equilibria with full coverage" *characterized by* $x_1 = x_2 = 1, y_1 = y_2 = 0$ *and* $p_i \in [-u, u]$, $p_{-i} = -p_i$ *for i=1,2*

Proof: see appendix.

Despite the restrictions that deviations which undercut price on one side and raise the price on the other side impose, market shutdown is compatible with subgame perfect Nash equilibrium of the price competition game, and so is firms getting positive profit. The next theorem shows that these type of equilibria are not compatible with coalitional rationalizability. Theorem 7 implies that in every crspe of the price competition game both firms get zero profit and that there is full market participation. These results are again analogous to the ones on one-sided markets with externalities. In particular, if consumers are homogeneous on both sides, then in two-sided markets with network externalities the assumption of coalitional rationalizability reestablishes that Bertrand competition drives firms' profits down to zero. On the

[4] This type of strategy, known in the literature as "Divide and Conquer", was first analyzed by Jullien (2001) and Caillaud-Jullien (2001a).

other hand, prices do not have to be zero, it is compatible with equilibrium that one side of the market is subsidized, while consumers on the other side pay a strictly positive price.

Theorem 7 *There are three types of crspe of the game of price competition between two firms on a two-sided market with symmetric, homogeneous network externalities. The first is characterized by* $x_1 = x_2 = 1, y_1 = y_2 = 0$ *and* $p_1 = -p_2$, $p_1 \in [-u, u]$. *The second is characterized by* $x_1 = x_2 = 0, y_1 = y_2 = 1$ *and* $q_1 = -q_2$, $q_1 \in [-u, u]$. *The third is characterized by* $x_1 = x_2 = y_1 = y_2 = \frac{1}{2}$, $p_1 = q_1 = -p_2 = -q_2 \in [-u/2, u/2]$.

Proof: see appendix.

6 Two-sided Markets with Heterogeneous Consumers.

In this section, we drop the assumption that consumers on a two-sided market are homogeneous. For analytical tractability, the type of heterogeneity we introduce is of a particular form. We assume that on each side there are two types of consumers: one type has a higher intensity parameter than the other. We show that even this simple departure from the model with homogeneous consumers can substantially change the nature of optimal pricing and price competition on two-sided markets with network externalities.

6.1 Monopoly

This section investigates a monopolist network provider's pricing decisions on a two sided market with network externalities, in which consumers are heterogeneous on each side of the market.

Formally, consider the extensive form game which has the same players and moves as the game described in Section 5.1 and in which the firm's payoff functions are the same, but consumers' payoffs differ the following manner. A fraction $a \in (0, 1)$ of the consumers on each side of the market have an intensity parameter h, while the remaining $1 - a$ fraction of the consumers on each side of the market have an intensity parameter l[5], where $l > 0$ and $h > l$. We call consumers with intensity parameter h high types and consumers with intensity parameter l low types. A high type side 1 consumer's utility from joining a network is $hx_2 - p_1$ where x_2 is the number

[5] The assumption that the fraction of high type consumers is the same on both sides is made to simplify the analysis.

of side 2 consumers joining the network, while a low type side 1 consumer's utility from joining the network is $lx_2 - p_1$. Similarly, a high type side 2 consumer's utility from joining a network is $hx_1 - p_2$ and a low type side 2 consumer's is $lx_1 - p_2$.

Theorem 8 below characterizes the set of crspe for different parameter values of the model and can be summarized as follows. If the intensity parameter of the low types is higher than a certain fraction k_1 of the intensity parameter of high types, then in all crspe the monopolist charges (l, l) and attracts all consumers. The higher the ratio of high types among consumers is, the higher the threshold fraction k_1 is. If the intensity parameter of low types is lower then fraction k_2 of the intensity parameter of high types, where $k_2 < k_1$, then the monopolist charges (ah, ah) and attracts all high type consumers. Again the higher the ratio of high types is the higher k_2 is. If the intensity parameter of low types relative to the intensity parameter of the high types is in between k_2 and k_1 then the firm charges h on one side of the market and attracts all high types there, and al on the other side of the market and attracts all consumers there.

In short, as one would expect, if the intensity parameter of low types is relatively small and there are not that many of them then the monopolist only targets high types. If the intensity parameter of low types is relatively high and there are a lot of them, then the monopolist targets all consumers. In cases in between the monopolist might target all consumers on one side and only the high types on the other side.

Theorem 8 *If $l > \frac{a}{2-a}h$ then in every crspe the firm charges (l, l) and all consumers on both sides join the network. If $l \in ((2a-1)h, \frac{a}{2-a}h)$ then in every crspe the firm charges h on one side and al on the other and every high type joins on the former side, while every consumer joins on the latter side. If $l < (2a-1)h$ then in every crspe the firm charges (ah, ah) and every high type consumer joins the network on both sides. If $l = \frac{a}{2-a}h$ then in crspe the firm can charge (l, l), (h, al) or (al, h). If $l = (2a-1)h$ then in crspe the firm can charge (ah, ah), (h, al) or (al, h).*

Proof: see appendix.

Note that if $a < 1/2$, then there is no crspe in which the monopolist charges a high price on both sides of the market, targeting only high types. Charging a high price on one side of the market has to be accompanied by charging a low price on the other side, so that there are enough consumers from the other side on the network for high types on the first side to be willing to pay the high price. The monopolist therefore cannot extract all consumer surplus from all high type consumers.

The above analysis assumed that the monopolist can only operate one network. The next theorem shows that, somewhat surprisingly, operating two networks but pricing them differently can improve the profit of the monopolist for certain parameter values. This is the case when there are relatively few high type consumers, but the high types' intensity parameter is very high compared to the low types. In this case, establishing a network which is cheap on side 1 and another one which is cheap on side 2 enables the monopolist to charge a high price on side 2 at the first network and still attract there all high types from that side, and in the meantime charge a high price on side 1 at the second network and still attract there all high types from that side. This way the monopolist can extract a high fraction of the consumer surplus from the high type consumers on both sides, something which he cannot do by operating only one network.

Theorem 9 *Suppose that the monopolist has the option to establish either one or two networks. If* $l \in \left((4a-1)h, \frac{a}{2-a}h\right)$*, or if* $l \in \left(\frac{a}{2-a}h, \frac{a(1-2a)}{1-a}h\right)$ *and* $a \in \left(0, 1-\frac{\sqrt{2}}{2}\right)$*, in any crspe he establishes two networks and charges* $p_1 = q_2 = h(1-2a) + al$*,* $p_2 = q_1 = al$*. If* $l = (4a-1)h$*, then in crspe the firm can either establish one network and charge* (h, al) *or* (al, h) *or establish two networks and charge* $p_1 = q_2 = h(1-2a) + al$*,* $p_2 = q_1 = al$*. If* $l = \frac{a(1-2a)}{1-a}h$ *and* $a \in \left(0, 1-\frac{\sqrt{2}}{2}\right)$ *then in crspe the firm can either establish one network and charge* (l, l) *or establish two networks and charge* $p_1 = q_2 = h(1-2a) + al$*,* $p_2 = q_1 = al$*.*

Proof: see appendix.

When establishing two networks, the monopolist has to satisfy certain constraints in price setting. For a low type consumer, joining the network that is cheaper to him has to yield nonnegative utility in equilibrium. Similarly, for a high type consumer, joining the network that is more expensive to him in equilibrium has to yield nonnegative utility. We refer to these as individual rationality constraints. Also, for low type consumers, joining the cheaper network in equilibrium has to yield at least as much utility as joining the more expensive network. Similarly, for high types joining the more expensive network in equilibrium has to yield at least as much utility as joining the cheaper network. We refer to these as incentive compatibility conditions. As usual, the individual rationality constraints for the high types and the incentive compatibility constraints for the low types are binding in equilibrium. Finally, equilibrium pricing has to satisfy coalitional rationalizability constraints meaning that no group of consumers can have a supported restriction that invalidates the equilibrium. In particular, one network cannot be more expensive than the other on both sides of the market, otherwise the coalition of all consumers have a supported restriction not to join the more expensive network.

The result in theorem 9 is counterintuitive because by establishing two active networks the monopolist is splitting the consumers and therefore sacrifices gross consumer surplus. And the maximum price the monopolist can extract from a given consumer is increasing in the gross consumer surplus of that consumer in equilibrium. The intuition is that if the consumer surplus that is sacrificed this way is small, then establishing two networks can be profitable for the monopolist because he can extract a high fraction of the surplus from more consumers at the same time.

The result that crspe with one firm, two-sided market and heterogeneous consumers can be socially suboptimal is in contrast with the results on one-sided markets and two-sided markets with homogeneous consumers.

7 Conclusions and future directions

We investigated the consequences of the assumption that consumers on markets with network externalities are capable of getting around coordination problems as long as it does not require explicit communication. We showed that if the market is either one-sided or if it is two-sided but consumers on each side are homogeneous, then the assumption reestablishes classic results on price setting by a monopolist and on Bertrand competition. We showed that allowing for heterogeneity with respect to how much consumers are affected by the network externality might induce a monopolist to establish multiple networks to price discriminate consumers more effectively.

We are currently working on the analysis of Bertrand competition on two-sided markets with network externality and heterogeneous consumers. Preliminary results indicate that there can be coalitionally rational equilibria of this pricing game in which firms get positive profits. The network structure in these equilibria is similar to the structure that the monopolist establishes if it is optimal for him to establish multiple networks. In particular if a network is relatively cheap for one side of the market, then it has a relatively large number of consumers on that side, and if a network is relatively cheap on one side, then it is relatively expensive on the other side.

Other extensions of the model that are in progress include the analysis of two-sided markets with asymmetric sides and the introduction of usage fees (a fee that a consumer has to pay only if he found a successful match on the network) besides registration fees.

8 Appendix.

Proof of Theorem 1:

Let p be the price that the firm charges in s. It has to be that $p > 0$ otherwise all consumers join the network and the firm gets negative profit which cannot be in equilibrium. Let C be the set of consumers joining the network in s and let $c = \int_{i \in C} dF(i)$, the number of consumers joining the network in equilibrium. Then for every consumer $i \in C$ it holds that $u_i c - p > 0$. Then for every $\varepsilon > 0$ and $i \in C$ it holds that $u_i c - p + \varepsilon > 0$, which implies that after a price announcement of $p - \varepsilon$ joining the network is a supported restriction for C. That means that if consumers are coalitionally rational, then the firm can guarantee a profit arbitrarily close to π by charging a price of $p - \varepsilon$ where $\varepsilon > 0$ is small enough. But that implies that it cannot be that $\pi' < \pi$. For the second claim note that by assumption $\exists\ e > 0$ such that $F(e) < 1$. Let E denote the set of consumers with higher intensity parameter than e. Then $i \in E$ implies $u_i - e > 0$ and therefore $u_i(1 - F(e)) - e(1 - F(e)) > 0$. This means that after a price $e(1 - F(e))$ for the coalition of consumers who have intensity parameter higher than e it is a supported restriction to join the network, guaranteeing a profit of at least $F(e)e(1 - F(e)) > 0$. QED

Proof of Theorem 2:

For any $\varepsilon > 0$ it holds that after a price announcement of $b - \varepsilon$ the coalition of all consumers has a supported restriction to join the network. Therefore, if consumers are coalitionally rational then the firm could get a profit arbitrarily close to b by setting a price $b - \varepsilon$ for small enough $\varepsilon > 0$. But then in no crspe the firm's profit can be less than b. In any subgame perfect Nash equilibrium no consumer joins the network after a price announcement larger than b. But then the only candidate for crspe is if the firm sets a price b and all but a measure zero of consumers join the network. After a price announcement of b it is straightforward to show that no coalition of consumers have a nontrivial supported restriction and it is a Nash equilibrium of this subgame that all but a measure zero of consumers join the network. If the price is smaller than b then the only coalitionally rationalizable profile is all consumers joining the network and if the price is larger than b then the only coalitionally rationalizable profile is no consumers joining the network. This concludes that the proposed profile is indeed a crspe. QED

Theorem 10 *The set of spne of the game of price competition between two firms on a one-sided market with network externalities and homogeneous consumers is the following:*

1) "Pessimistic equilibria" *characterized by* $x = y = 0$, $p \geq 0, q \geq 0$

2) "Monopolistic equilibria with full coverage" *characterized by* $x = 1$, $y = 0$, $p = \hat{p} \in [0, u]$, $q \geq p - u$ *(or, analogously, by* $y = 1, x = 0$, $q = \hat{p} \in [0, u]$, $p \geq q - u$*)*

3) "Monopolistic equilibria with partial coverage" *characterized by* $0 < x < 1$, $y = 0$, $p = xu$, $q \geq 0$ *(or, analogously, by* $0 < y < 1, x = 0$, $q = yu$, $p \geq 0$*)*

4) "Symmetric equilibria with full coverage" *characterized by* $x = y = \frac{1}{2}$ *and* $p = q = \hat{p} \in \left[0, \frac{u}{2}\right]$

5) "Symmetric equilibria with partial coverage" *characterized by* $x = y = \hat{x} \in \left(0, \frac{1}{2}\right)$ *and* $p = q = \hat{x}u$

6) "Asymmetric equilibria" *characterized by* $x > y > 0$ *(or, analogously, by* $y > x > 0$*) and* (p, q) *such that* $xu - p = yu - q \geq 0$

Proof.

If an equilibrium exists, it has to be the case that in equilibrium market shares satisfy one of the following conditions:

1. $x = y = 0$
2. $x = 1$, $y = 0$ or, analogously, $y = 1, x = 0$
3. $0 < x < 1$, $y = 0$ or, analogously, $0 < y < 1, x = 0$
4. $x = y = \frac{1}{2}$
5. $x = y = \hat{x} \in \left(0, \frac{1}{2}\right)$
6. $x > y > 0$ or, analogously, $y > x > 0$.

We will now show that in each of these six cases either an equilibrium exists and it is one of those listed in the theorem, or it does not exist.

1. $x = y = 0$.
First, observe that if such an equilibrium exists it has to be the case that $p \geqslant 0, q \geqslant 0$, otherwise consumers of at least one side would have an incentive to enter the market and collect the subsidy regardless of the network size. Denote with æ (p, q) and ǥ (p, q) the expected market share of network A and network B respectively, after price (p, q) are announced. Let æ $(p, q) = 0\ \forall\ (p, q)$ such that $p \geqslant 0$ and ǥ $(p, q) = 0$, $\forall$ (p, q) such that $q \geqslant 0$ .These expectations are self-fulfilling because if consumers expect that a given network, which charges a nonnegative price, will be empty, then joining that network is weakly dominated by the choice to stay out of the market. Given these expectations, if both networks charge nonnegative prices they collect zero profits and they have no profitable deviation because the only possible way for a network to attract a positive amount of consumers is to charge a nonpositive price thus collecting nonpositive profits. Therefore, there exist equilibria with $x = y = 0$ and $p \geqslant 0, q \geqslant 0$ and they are the only equilibria with zero coverage.

2. $x = 1$, $y = 0$, (the proof is analogous for $y = 1, x = 0$).
First, observe that if such an equilibrium exists it has to be the case that $p \leqslant u$ because otherwise consumers would rather stay out of the market then join network A, and $q \geqslant p - u$, otherwise consumers would rather choose B than A. Moreover, it has to be $p \geqslant 0$ otherwise A would lose money. Therefore, if such an equilibrium exists, it has to be such that $p =$ þ $\in [0, u]$ and $q \geqslant p - u$. Now, let us construct expectations to sustain these equilibria. For a given þ $\in [0, u]$, let æ $(p, q) = 1$ and ǥ $(p, q) = 0\ \forall\ (p, q)$ such that $p \leqslant$ þ and $q \geqslant$ þ$-u$. These expectations are self-fulfilling because Ɵ$(A) = u-$þ $\geqslant 0 \geqslant -q =$ Ɵ(B). Moreover, let æ $(p, q) =$ ǥ $(p, q) = 0\ \forall\ (p, q)$ such that $p >$ æ and $q \geqslant 0$. This further restriction is also self-fulfilling. Given these expectations, there is no profitable deviation for either A or B from the suggested strategies. In particular, B collects zero profits and has no incentive to deviate to a different price because he could get a positive market share only with a negative price. As for A, she collects nonnegative profits (actually, positive profits, except for the case of þ $= 0$) and would collect nonpositive profits after any deviation. Therefore this is an equilibrium.

3. $0 < x < 1$, $y = 0$ (the proof is analogous for $0 < y < 1, x = 0$)
If such an equilibrium exists, it has to be the case that consumers

are indifferent between joining A or staying out and prefer A to B, which implies $\theta(A) = ux - p = 0$ and $\theta(A) = 0 > -q = \theta(B)$ which in turn imply $p = ux$ and $q > 0$. Now, we have to construct expectations that sustain this equilibrium. When consumers observe (p, q) such that $p = ux$ and $q > 0$, let $x^e(p, q) = x$ and $y^e(p, q) = 0$ and let a proportion x of the consumers join A and the remaining fraction stay out. Moreover, let $y^e(p, q) = 0$ in every subgame where $q > 0$ and let $x^e(p, q) = 0 \;\forall\; (p, q)$ such that $p > u$. This further restriction is also self-fulfilling. Given these expectations, there is no profitable deviation for either A or B from the suggested strategies, therefore this is an equilibrium.

4. $x = y = \frac{1}{2}$.
   First, if such an equilibrium exists it has to be the case that $p = q = \bar{p} \in \left[0, \frac{u}{2}\right]$ otherwise consumers would not be optimizing. Fix a number $\bar{p} \in \left[0, \frac{u}{2}\right]$ and let $x^e = y^e = \frac{1}{2} \;\forall\; (p, q)$ such that $p = q = \bar{p}$ and $x^e = y^e = 0$ after every other history of nonnegative prices. Given these expectations, if firms announce $p = q = \bar{p}$ consumers are actually indifferent and so the expectation that they equally split between the networks is self-fulfilling. Moreover, it is optimal for the firms to charge $p = q = \bar{p}$ because every deviation would give nonpositive profits.

5. $x = y = \alpha \in \left(0, \frac{1}{2}\right]$
   First, if such an equilibrium exists it has to be the case that $p = q = \bar{p} \in \left(0, \frac{u}{2}\right]$ otherwise consumers would not be optimizing. Now, fix a number $\alpha \in \left(0, \frac{1}{2}\right]$ and let expectations be such that when consumers observe prices $p = q = \alpha u$ they expect $x^e = y^e = \alpha$ and a fraction α chooses A, a fraction α chooses B and the remaining fraction, $1 - 2\alpha$, stays out. Moreover, let $x^e = y^e = 0$ after every other history of nonnegative prices. These expectations are self-fulfilling and prevent any profitable deviation from the candidate equilibrium $x = y = \alpha$, $p = q = \alpha u$.

6. $x > y > 0$ (the proof is analogous for $y > x > 0$)
   Fix $(x, y)^*$ such that $x^* > y^* > 0$. If such an equilibrium exists, it has to be such that prices make consumers indifferent between the two networks, i.e. $(p, q)^* \equiv (p, q)$ such that $ux^* - p = uy^* - q > 0$. Let expectations be such that if $(p, q) = (p, q)^*$, then $x(p, q) = x^*$ and $y(p, q) = y^*$, if $p > 0$ and $p \neq (p)^*$ then $x(p, q) = 0$ and if $q > 0$ and $q \neq (q)^*$ then $y(p, q) = 0$. These expectations are self-fulfilling and firms will charge $(p, q)^* \equiv (p, q)$ because this strategy profile guaran-

tees that profits are strictly positive while every deviation would imply nonpositive profits. QED

Proof of Theorem 3:

By assumption there is $e > 0$ such that $F(e) < 1$. Let E be the set of consumers whose intensity parameter is higher than e. Then for every $p \in (0, e(1-F(e))$ it holds that $u_i(1-F(e))-p > 0 \; \forall \, i \in E$ (joining together a network that charges p is strictly better for everyone in E than not joining any network).

Assume now that one firm charges p, the other charges q and $p < q$. We claim that if there is a crspe like that, then no consumer joins the network charging q. Let R_q be the set of consumers to whom it is rationalizable to join the extensive network in the subgame where prices are p and q. If $R_q = \emptyset$ then since coalitional rationalizability implies rationalizability, no consumer joins the more expensive network in any crspe. Let now $R_q \neq \emptyset$ and let $r_q = \int_{i \in R_q} dF(i)$, the number of consumers in R_q. Then for every $i \in R_q$ the maximum utility i can expect when joining the more expensive network is $u_i r_q - q$, which has to be nonnegative, otherwise $i \notin R_q$. The minimum utility that i can expect if every consumer in R_q joins the cheaper network is $u_i r_q - p$. But $u_i r_q - p > u_i r_q - q$ and therefore $u_i r_q - p > 0$, which implies that joining the cheaper network is a supported restriction for R_q. Therefore joining the more expensive network is not coalitionally rationalizable for any consumer. This implies that the firm charging a higher price gets zero profit in any crspe. It cannot be that in a crspe $p < 0$ because then in this equilibrium all consumers have to join the firm charging p and therefore this firm gets a negative profit. It cannot be that $p = 0$ because then the firm charging p gets 0 profit, while coalitional rationalizability implies that deviating to $\min(e(1-F(e))/2, q/2)$ yields him a positive profit. This is because $\min(e(1-F(e))/2, q/2) < e(1-F(e))/2$ and $\min(e(1-F(e))/2, q/2) < q$ and therefore it is a supported restriction for consumers in E to join the firm if charging $\min(e(1-F(e))/2, q/2)$, which implies that the firm gets positive profit since $\min(e(1-F(e))/2, q/2) > 0$. It cannot be that $p > 0$ and that no consumers join the firm charging p because then this firm could again profitably deviate to $\min(e(1-F(e))/2, q/2)$. Therefore it has to be that $p > 0$ and the number of consumers joining the firm charging p is $x > 0$ while the number of consumers charging q is 0. But then the firm charging q in equilibrium (and getting zero profit) could profitably deviate to $p/2 > 0$ since then at least $x > 0$ consumers would join its network. This concludes that in crspe it cannot be that the two networks charge different prices.

Moreover, it cannot be that $p = q < 0$ because then at least one of the firms would get negative profit. Assume now that $p = q > 0$. If $x + y = 0$ then firms get 0 profit, but each firm could get positive profit by deviating to $\min(e(1 - F(e))/2, p/2)$. If $x + y > 0$ then at least one of the firms gets no more than $p \cdot (x + y)/2$, but it could get a profit that is no less than something arbitrarily close to $p \cdot (x + y)$ by deviating to $p - \varepsilon$ for small enough $\varepsilon > 0$. Therefore there is no crspe of this kind either and only prices $p = q = 0$ can be part of an equilibrium. Let x denote the number of consumers joining A and y the number of consumers joining B. Then only the following market shares are compatible with equilibrium: all consumers with positive u_i parameter sign up for firm A and $x \geq y$; all consumers with positive u_i parameter sign up for firm B and $x \leq y$; all consumers with positive u_i parameter sign up for one of the networks and $x = y$; and finally that no consumer joins any network. The latter is not compatible with coalitional rationalizability, since the restriction "join either A or B" is supported for the set of consumers with positive intensity parameter. Let the number of consumers in this set be a. Note that by assumption $a > 0$ and therefore the restriction guarantees an expected payoff of at least $u_i a/2 > 0$ for every consumer i belonging to this coalition, while staying out gives 0 utility.

All that remains to show is that for each of the remaining cases there is a crspe. Let s be a strategy profile for the consumers in the subgame after $(0, 0)$ prices that gives rise to one of the above market shares. Note that if the two networks charge the same price then if joining A is coalitionally rationalizable for a consumer, then so is joining B. If these two prices are zero, then for consumers with zero intensity parameter all strategies are coalitionally rationalizable, while for consumers with positive parameter the strategies joining A and joining B are. But then s is coalitionally rationalizable in the subgame. Then consider any profile in which firms charge zero prices and consumers play s in the subgame following $(0, 0)$, all consumers join B in subgames after $(p, 0)$ for $p > 0$, all consumers join A in subgames after $(0, q)$ for $q > 0$, and consumers play an arbitrary coalitionally rationalizable equilibrium in every other subgame. Then in every subgame consumers play a coalitionally rationalizable equilibrium and no firm has a profitable deviation and therefore this profile is crspe. QED

Proof of Theorem 4:

If an equilibrium exists, it has to be the case that in equilibrium market shares satisfy one of the following conditions:

1. $x_1 = x_2 = 0$
2. $x_1 = x_2 = 1$
3. $0 < x_1 + x_2 < 2$

We will now show that in each of these three cases either an equilibrium exists and it is one of those listed in the theorem, or it does not exist.

$x_1 = x_2 = 0.$

We will show that there is no such equilibrium. Suppose prices are p_1, p_2 and every consumer is out of the market in equilibrium. Then $p_1, p_2 \geqslant 0$, otherwise if the firm offered at least one negative price some consumers would rather subscribe than stay out. Suppose the monopolist deviates and offers $(p_1, p_2) = (-\varepsilon, u - \varepsilon)$. All consumers of type 1 would subscribe for sure so that they can collect the subsidy and therefore all consumers of type 2 would subscribe as well, because their utility would be $U = u(1) - u + \varepsilon > 0 = U(out)$. This deviation would give the monopolist profits $\pi = u > 0$, therefore there is no such equilibrium.

$x_1 = x_2 = 1$

Let $x_1^e(p_1, p_2), x_2^e(p_1, p_2)$ denote expected market shares after prices (p_1, p_2) are announced and let $x_1^e(p_1, p_2) = x_2^e(p_1, p_2) = 1 \forall (p_1, p_2)$ such that $p_1, p_2 \leqslant u$. These expectations are self-fulfilling and allow the monopolist to charge $(p_1, p_2) = (u, u)$ extracting the maximum possible profit, $\pi = 2u$. Let expectations be such that $x_1^e(p_1, p_2) = x_2^e(p_1, p_2) = 0 \forall (p_1, p_2)$ such that $p_1, p_2 > 0$ and $x_i^e(p_1, p_2) = 1 \forall (p_1, p_2)$ such that $p_i \leqslant 0$. These expectations are self-fulfilling and the monopolist can charge $(p_1, p_2) = (0, u)$ thus inducing all consumers to subscribe and receiving profits equal to u. There is no better strategy for the monopolist because if he increases at least one price he obtains zero market shares and zero profits while if he decreases at least one of the prices he keeps the same market shares but his profits decrease. Let expectations be such that $x_1^e(p_1, p_2) = x_2^e(p_1, p_2) = 1 \forall (p_1, p_2)$ such that $p_i \leqslant \bar{p}_i \leqslant u$ for $i = 1, 2$, $x_1^e(p_1, p_2) = x_2^e(p_1, p_2) = 0 \forall (p_1, p_2)$ such that $p_i > \bar{p}_i$ for $i = 1, 2$ and for any other couple of nonnegative prices. If both prices are nonnegative, these expectations are self-fulfilling and allow the monopolist to charge $(p_1, p_2) = (\bar{p}_1, \bar{p}_2)$ thus receiving profits $\bar{p}_1 + \bar{p}_2$.

The monopolist has no incentive to increase both prices because he would get zero market shares. He has no incentive to decrease both prices (or just one, keeping the other constant) because his market shares would be constant but his profit would decrease. No other couple of nonnegative profits would give a positive profit. Similarly for any couple of nonpositive prices. Finally, he might charge $(p_1, p_2) = (0, u)$ and receive profits $\pi = u$. Therefore, an equilibrium $(p_1, p_2) = (\hat{p}_1, \hat{p}_2)$ exists only if $\hat{p}_1 + \hat{p}_2 > u$. We have shown that the level of profits that the monopolist can achieve in equilibria with full coverage is in the range $[u, 2u]$.

$0 < x_1 + x_2 < 2$

Fix $(\hat{p}_1, \hat{p}_2)$ such that $\hat{p}_1, \hat{p}_2 \in (0, u)$ and let expectations be such that $x^e_i(\hat{p}_1, \hat{p}_2) = \frac{\hat{p}_i}{u}$ for $i = 1, 2$ and $x^e_1(p_1, p_2) = x^e_2(p_1, p_2) = 0$ for any other couple of nonnegative prices. These expectations are self-fulfilling because if consumers observe $(\hat{p}_1, \hat{p}_2)$ and have these expectations they are indifferent between joining the network or staying out, so it is possible that exactly a fraction $x^e_i(\hat{p}_1, \hat{p}_2)$ enters the market and $x_i = \frac{\hat{p}_i}{u}$ for $i = 1, 2$. If $\hat{p}_1 x_1 + \hat{p}_2 x_2 > u$ the monopolist has no better strategy than charging $(p_1, p_2) = (\hat{p}_1, \hat{p}_2)$. therefore this is an equilibrium. QED.

Proof of Theorem 5:

Suppose consumers are coalitionally rational. Then, in every subgame where they observe (p_1, p_2) such that $p_i < u$ for $i = 1, 2$ there is a supported restriction for the coalition of all consumers on both sides which consists in excluding the strategy "stay out of the market". Therefore, the only Nash equilibrium for these subgames which is also compatible with coalitional rationality is if all consumers join the network. Therefore the monopolist can get a profit arbitrarily close to $2u$ if consumers are coalitionally rational and therefore its profit in crspe is at least $2u$. Since $2u$ is the maximum possible consumer surplus and consumers cannot get negative utility in equilibrium, the monopolist's profit has to be $2u$ in every crspe. Then since $p_i > u$ $(i = 1, 2)$ implies that no side i consumer join the network, it has to be that in every crspe the firm sets prices (u, u) and every consumer on both sides joins the network (with the exception of a set with measure zero). QED

Proof of Theorem 6:

If an equilibrium exists, it has to be the case that in equilibrium market shares satisfy one of the following conditions:

1. $x_1 = y_1 = x_2 = y_2 = 0$

2. $x_1 = x_2 = y_1 = y_2 = \frac{1}{2}$

3. $x_1 = y_1 < \frac{1}{2}$ and $x_2 = y_2 < \frac{1}{2}$

4. $x_i = y_i < \frac{1}{2}$and $x_{-i} = y_{-i} = \frac{1}{2}$, $i = 1, 2$, $i \neq -i$

5. $x_1 = x_2 = 1, y_1 = y_2 = 0$

6. $x_1, x_2 > 0$, $y_1 = y_2 = 0$ and $x_1 + x_2 < 2$.

7. $x_1 \geq y_1 > 0$ and $x_2 > y_2 > 0$

8. $0 < x_i < y_i$ and $x_{-i} > y_{-i} > 0$, $i = 1, 2$, $i \neq -i$

We will now show that in each of these eight cases either an equilibrium exists and it is one of those listed in the theorem, or it does not exist. First of all, a general observation: for a given strategy profile to be an equilibrium the following necessary conditions have to be satisfied:

$$\pi^a \geq 0 \quad (1^*)$$
$$\pi^b \geq 0 \quad (2^*)$$
$$q_1 + \inf\{q_2, 0\} \leq \pi^a \quad (3^*)$$
$$q_2 + \inf\{q_1, 0\} \leq \pi^a \quad (4^*)$$
$$p_1 + \inf\{p_2, 0\} \leq \pi^b \quad (5^*)$$
$$p_2 + \inf\{p_1, 0\} \leq \pi^b \quad (6^*)$$
$$x_i > 0 \Rightarrow p_i \leq x_{-i}u \text{ for } i = 1, 2 \quad (7^*)$$
$$y_i > 0 \Rightarrow q_i \leq y_{-i}u \text{ for } i = 1, 2 \quad (8^*)$$

The first two conditions require that no firm gets negative profits, the last two require that if some consumers of a given type subscribe for one of the firms they receive at least their reservation utility of zero, and conditions (3*) to (6*) require that no firms finds it profitable to follow one of the so called "divide and conquer" deviations. The purpose of a DC deviation is to get market share equal to one on both sides and the way the result is obtained is by offering a very convenient price to one group of consumers, so that it becomes strictly dominant for them to join the deviating firm, and then charge a high price to consumers on the other side, who are willing to pay it because they are sure that the deviating firm has a large network. More precisely, suppose firms are charging prices p_1, p_2, q_1, q_2. Firm B could follow a DC deviation charging $q_1 < p_1 - u$ to type 1. Consumers of type 1 would compare $U(A) = x_2 u - p_1 \leq u(1) - p_1$ and $U(B) = y_2 u - q_1 > u(0) + u - p_1$ and thus choose B. Given this fact, B could charge $q_2 = u + \inf\{p_2, 0\}$ and thus attract all 2 users. Equation (3*) says that the total profit from this deviation, which is $q_1 + q_2 < p_1 - u + u + \inf\{p_2, 0\} = p_1 + \inf\{p_2, 0\}$ cannot exceed equilibrium profits.

$x_1 = y_1 = x_2 = y_2 = 0$

Suppose prices are p_1, p_2, q_1, q_2 and every consumer is out of the market in equilibrium. Then $p_1, p_2, q_1, q_2 \geq 0$, otherwise if at least one firm offered a subsidy some consumers would rather subscribe with that firm than stay out, even if no consumer of the other type subscribed with that firm. This implies $\pi^a = \pi^b = 0$. Conditions (1*) to (6*) can be rewritten as:

$$\begin{aligned} \pi^a &\geq 0 && (1^{**}) \\ \pi^b &\geq 0 && (2^{**}) \\ q_1 &\leq 0 && (3^{**}) \\ q_2 &\leq 0 && (4^{**}) \\ p_1 &\leq 0 && (5^{**}) \\ p_2 &\leq 0 && (6^{**}) \end{aligned}$$

They are all satisfied only if prices are all zero, therefore the only candidate equilibrium with zero participation has zero prices. Now, we to show that this actually is an equilibrium; we already took care of the DC deviations, which consist in charging $\left(p_i^d, p_j^d\right) = (-\varepsilon - u, u)$. First, notice that since these deviations are unprofitable, then any other deviation where prices are smaller or equal than these cannot be profitable. Now, let expectations be such that if a firm, say firm A, deviates to (p_1, p_2) with $p_1, p_2 \geq -u$ then every consumer on both sides joins the other firm, so that the deviation profit is zero. Moreover, let expectations be such that if a firm, say firm A again, deviates to $p_i < -u$ and $p_{-i} > u$ then every consumer on side i joins A and every consumer on side $-i$ stays out, which gives A negative profits. It is easy to verify that the expectations we specified are self-fulfilling and therefore there exists no profitable deviation from the candidate equilibrium.

$x_1 = x_2 = y_1 = y_2 = \frac{1}{2}$

First, notice that if such an equilibrium exists it has to be the case that on each side prices are identical ($p_1 = q_1$ and $p_2 = q_2$)otherwise, from the point of view of each group, the cheaper network would be strictly preferred. Now, denoting with (p_1, p_2) the prices charged by both firms, conditions (1*) to (8*) can be rewritten as[6]:

$$
\begin{aligned}
\frac{1}{2}(p_1 + p_2) &\geq 0 && (1^{***}) \\
p_1 + \inf\{p_2, 0\} &\leq \frac{1}{2}(p_1 + p_2) && (3^{***}) \\
p_2 + \inf\{p_1, 0\} &\leq \frac{1}{2}(p_1 + p_2) && (4^{***}) \\
p_1 &\leq \frac{u}{2} && (7^{***}) \\
p_2 &\leq \frac{u}{2} && (8^{***})
\end{aligned}
$$

¿From (1***), either both prices are nonnegative or one is nonpositive and one is nonnegative. First, suppose that both prices are nonnegative and such that $p_i \in \left[0, \frac{u}{2}\right]$, $i = 1, 2$. Then (1***), (7***) and (8***) are satisfied. From (3***) and (4***) we get

$$
\begin{aligned}
p_1 &\leq \frac{1}{2}(p_1 + p_2) && (1) \\
p_2 &\leq \frac{1}{2}(p_1 + p_2) && (2)
\end{aligned}
$$

which imply $p_1 = p_2$. Now, we have constructed candidate equilibria with $p_1 = p_2 = p \in \left[0, \frac{u}{2}\right]$, where firms make nonnegative profits $\pi \in \left[0, \frac{u}{2}\right]$ and we need to show that firms have no profitable deviations from these strategy profile. Suppose one firm sets prices (p, p) and the other, say firm A, deviates to $(p_1, p_2) \neq (p, p)$. If $p_1 < p - u$ and $p_2 < u$ or if $p_1 < u$ and $p_2 < p - u$ this is a DC deviation, which is not profitable in our candidate equilibria. If $p_i < p - u$ and $p_{-i} \geq u$ for $i = 1, 2$, let expectations be such that i-consumers choose A and $-i$-consumers stay out . Finally, let expectations be such that if $p_1, p_2 > p - u$ every consumer of every type chooses B. These expectations are self-fulfilling and imply negative deviation profits. Therefore there exist equilibria with $p_1 = p_2 = p \in \left[0, \frac{u}{2}\right]$ where firms make nonnegative profits $\pi \in \left[0, \frac{u}{2}\right]$ and there is no other equilibrium with these market shares and all nonnegative prices.

Let us now consider candidate equilibria with market shares equal to one half where the price charged to one side is positive and the other is negative. Let $p_i \leq 0$. Then (4***) implies $(p_i + p_{-i}) \leq \frac{1}{2}(p_i + p_{-i})$, which

[6] We omit (2*) (5*) and (6*) because if firms charge identical prices they are superfluous.

implies $(p_i + p_{-i}) \leq 0 \Rightarrow \pi \leq 0$.which, together with (1***) implies $\pi = 0$. This in turn implies $p_i = -p_{-i}$. Finally, from (8***) we get the further restriction that $p_{-i} \in \left[0, \frac{u}{2}\right]$. So, we have constructed candidate equilibria with market shares equal to one half, $p_i = -p_{-i}$ and $p_{-i} \in \left[0, \frac{u}{2}\right]$. To show that firms have no profitable deviations, let (p_i, p_{-i}) be the equilibrium prices and assume firm B deviates to (q_i, q_{-i}). If $q_i < p_i - u$ and $q_{-i} < u$ this is a DC deviation and it is not profitable by construction. If $q_{-i} < p_{-i} - u$ and $q_i < u + p_i$ this is another DC deviation and again it is not profitable. Let expectations be such that if $q_i < p_i - u$ and $q_{-i} \geq u$ all consumers on side i join B and all consumers on side $-i$ stay out, which gives B negative deviation profits. Analogously, let expectations be such that if $q_{-i} < p_{-i} - u$ and $q_i > u + p_i$ all consumers on side $-i$ join B and all all consumers on side i choose A, which gives B negative deviation profits. Finally, let expectations be such that if both $q_i \geq p_i - u$ and $q_{-i} \geq p_{-i} - u$ then all consumers on both sides choose A, so that B gets zero deviation profits.

$x_1 = y_1 < \frac{1}{2}$ and $x_2 = y_2 < \frac{1}{2}$

We will show that there is no such an equilibrium. First of all, prices have to be identical across firms on each side, since market shares are identical across firms. Moreover, all prices have to be nonnegative, otherwise every consumer who is out of the market would rather enter and receive the subsidy. Finally, since consumers on each side have to be indifferent between being in or out of the market prices have to be such that:

$$p_1 = q_1 = x_2 u > 0 \tag{3}$$

$$p_2 = q_2 = x_1 u > 0 \tag{4}$$

Now, consider the following DC deviation for firm B:

$$q_1 = p_1 - u - \varepsilon < p_1 - u \Rightarrow q_1 < x_2 u - u \Rightarrow U_1(B) > 0 \tag{5}$$

$$q_2 = u - \varepsilon < u \tag{6}$$

This gives strictly positive deviation profits, therefore this is not an equilibrium.

$x_i = y_i < \frac{1}{2}$and $x_{-i} = y_{-i} = \frac{1}{2}$

We will show that there is no such an equilibrium. First of all, prices have to be identical across firms on each side, since market shares are identical across firms. Moreover, prices on side i have to be nonnegative, otherwise every consumer who is out of the market would rather enter and receive the subsidy. Finally, since consumers on side i have to be indifferent between being in or out of the market and consumers on side $-i$ have to be willing to be in the market prices have to be such that:

$$p_i = q_i = \frac{u}{2} \tag{7}$$

$$p_{-i} = q_{-i} \leqslant x_i u < \frac{u}{2} \tag{8}$$

Now, consider the following DC deviation for firm B:

$$q_i = p_i - u - \varepsilon < p_i - u \Rightarrow q_1 < \frac{u}{2} - u \tag{9}$$

$$q_{-i} = u + \min\{p_{-i}, 0\} - \varepsilon < u + \min\{p_{-i}, 0\} \tag{10}$$

This gives deviation profits

$$\pi^d = \frac{u}{2} + \min\{p_{-i}, 0\} - 2\varepsilon < \frac{u}{2} + \min\{p_{-i}, 0\} \tag{11}$$

If $p_{-i} \geqslant 0$ this profit is positive and the deviation is profitable, hence there is no equilibrium with such market shares and all nonnegative prices. Suppose that, instead, $p_{-i} < 0$, we have

$$\pi^d = \frac{u}{2} + p_{-i} - 2\varepsilon < \frac{u}{2} + p_{-i}, \tag{12}$$

and we can show that this is positive as well, i.e. that

$$\frac{u}{2} + p_{-i} > 0 \Leftrightarrow p_{-i} > -\frac{u}{2} \tag{13}$$

because the following implications hold:

$$\pi^e = x_i p_i + p_{-i}\frac{1}{2} \geqslant 0 \Leftrightarrow \tag{14}$$

$$\Leftrightarrow x_i \frac{u}{2} + p_{-i}\frac{1}{2} \geqslant 0 \Rightarrow \tag{15}$$

$$\Rightarrow p_{-i} \geqslant -x_i u > -\frac{u}{2} \tag{16}$$

$x_1 = x_2 = 1, y_1 = y_2 = 0$

The necessary conditions (1*) to (8*) can be rewritten as[7]:

$$
\begin{array}{rcll}
(p_1 + p_2) & \geq & 0 & (1^{****}) \\
p_1 + \inf\{p_2, 0\} & \leq & \pi^b = 0 & (3^{****}) \\
p_2 + \inf\{p_1, 0\} & \leq & \pi^b = 0 & (4^{****}) \\
q_1 + \inf\{q_2, 0\} & \leq & \pi^a = (p_1 + p_2) & (5^{****}) \\
q_2 + \inf\{q_1, 0\} & \leq & \pi^a = (p_1 + p_2) & (6^{****}) \\
p_1 & \leq & u & (7^{****}) \\
p_2 & \leq & u & (8^{****})
\end{array}
$$

First, notice that the active firm cannot charge two strictly negative prices or one zero price and one strictly negative price in equilibrium because (1****) would be violated. Moreover, it cannot be the case that both prices charged by the active firm are strictly positive because this would contradict (3****) and (4****). Therefore, if an equilibrium with these market shares exists it has to be the case that the active form charges one nonnegative and one nonpositive price. Then, from either (3****) or (4****) we get $(p_1 + p_2) \leq 0$, which together with (1****) gives us $\pi = 0$. Therefore, we have $p_{-i} = -p_i \leq 0$ for either $i = 1$ or $i = 2$. From (7****) and (8****) we also get $p_i \in [0, u]$. Finally, it is necessary for this strategy profile to be an equilibrium that the inactive firm's prices (q_1, q_2) are such that $q_1 + q_2 \leq 0$ and $q_i \geq p - u$ and $q_{-i} \geq -p - u$. If $q_1 + q_2 > 0$, then firm A would have a DC deviation giving positive profits. If $q_i < p - u$ or $q_{-i} < -p - u$ then consumers on side 1 or side 2 are not in optimum when choosing firm A. Now that we have described the candidate equilibria which satisfy the necessary conditions we need to add sufficient restrictions on the expectations off-equilibrium path for these to be actual equilibria. Fix some p, q_1, q_2 like above. Consider the following set of self-fulfilling expectations. Let all consumers on both sides join A if prices are $(p, -p, q'_1, q'_2)$ and either $q'_1 \geq p - u$ or $q'_2 \geq -p - u$ or both. Let all consumers on both sides join B if prices are $(p, -p, q'_1, q'_2)$ and $q'_1 < p - u$ and $q'_2 < -p + u$ or if $q'_1 < u$ and $q'_2 < -p - u$. Let all consumers on side 1 join B and all consumers on side 2 stay out if $q'_1 < p - u$ and $q'_2 \geq -p + u$. Let all consumers on side 2 join B and all consumers on side 1 stay out if $q'_2 < -p - u$ and $q'_1 \geq u$. If prices are (p_1, p_2, q_1, q_2) then let consumers on either side choose A only if $p_1 = p$ and $p_2 = -p$ or if that is the only rationalizable choice for them. It is straightforward but tedious to explicitly provide such a profile. Then firm B can only get positive demand if he gets negative profits, while firm

[7] We skip condition (2*) because it is trivially satisfied.

A either sets its prices to be $(p, -p)$ or gets zero demand or negative profit. Therefore proposing prices $(p, -p, q_1', q_2')$ is consistent with equilibrium and then all consumers are in optimum by choosing A.

$x_1, x_2 > 0$, $y_1 = y_2 = 0$ and $x_1 + x_2 < 2$.

We will show that there is no such equilibrium. Suppose in equilibrium $(p_1, p_2, q_1, q_2, x_1, x_2)$, $x_1, x_2 > 0$ and $x_1 + x_2 < 2$. Then in equilibrium consumers on at least one side are indifferent between joining A and staying out. W.l.o.g. let this side be 1. Then it has to be that $p_1 > 0$. If $p_2 < 0$, consider the DC deviation by B $(q_1, q_2) = (p_1 - u - \varepsilon, p_2 + u - \varepsilon)$. It guarantees that every consumer joins B and for small enough ε a positive profit to B, since $p_1 + p_2 > x_1 p_1 + p_2 > 0$ (the latter is A's profit in the proposed equilibrium). If $p_2 > 0$, consider the DC deviation by B $(q_1, q_2) = (p_1 - u - \varepsilon, u - \varepsilon)$. It guarantees that every consumer joins B and for small enough ε a positive profit to B.

$x_1 \geq y_1 > 0$ and $x_2 > y_2 > 0$

We will show that there exists no such equilibrium. If it exists, it has to be the case that $p_1 > q_1$ and $p_2 > q_2$. Suppose $p_1 > p_2$. Then $p_1 > q_1$ and $p_1 > q_2$. Also $y_1 \leq 1/2$ and $y_2 < 1/2$ otherwise consumers who join A are not in optimum. Therefore $p_1 > q_1 y_1 + q_2 y_2$. Then if $p_2 > 0$ the DC deviation by B $(q_1, q_2) = (p_1 - u - \varepsilon, u - \varepsilon)$ guarantees that every consumer joins B and for small enough ε gives a profit larger than $q_1 y_1 + q_2 y_2$. If $p_2 < 0$, then $x_2 + y_2 = 1$. Then $p_1 + p_2 > p_1 x_1 + q_1 y_1 + p_2 x_2 + q_2 y_2 > q_1 y_1 + q_2 y_2$ (the latter holds because A's profits in the proposed equilibrium have to be nonnegative). Then if $p_2 < 0$ the DC deviation by B $(q_1, q_2) = (p_1 - u - \varepsilon, p_2 + u - \varepsilon)$ guarantees that every consumer joins B and for small enough ε gives a profit larger than $q_1 y_1 + q_2 y_2$. Similarly for the $p_1 < p_2$ case.

$0 < x_i < y_i$ and $x_{-i} > y_{-i} > 0$.

W.l.o.g. suppose $x_1 < y_1$ and $x_2 > y_2$. Then to make consumers indifferent between joining the two networks, $p_1 > q_1$ and $p_2 < q_2$. Suppose first that $p_1, p_2, q_1, q_2 > 0$. Then the DC deviation by B $(q_1, q_2) = (p_1 - u - \varepsilon, u - \varepsilon)$ guarantees that every consumer joins B and yields a profit of $p_1 - 2\varepsilon$. Also the DC deviation by A $(p_1, p_2) = (u - \varepsilon, q_2 - u - \varepsilon)$ guarantees that every consumer joins A and yields a profit of $q_2 - 2\varepsilon$. But since $p_1 + q_2 > p_1 x_1 + q_1 y_1 + p_2 x_2 + q_2 y_2$, for small enough ε it holds that

$p_1 - 2\varepsilon + q_2 - 2\varepsilon > p_1x_1 + q_1y_1 + p_2x_2 + q_2y_2$. Then either $p_1 - 2\varepsilon > q_1y_1 + q_2y_2$ or $q_2 - 2\varepsilon > p_1x_1 + p_2x_2$ (or both), establishing that at least one of the above DC deviations is profitable. Now assume that prices on one side of the market are negative. Wlog let $p_1 > q_1 > 0$ and $p_2 < q_2 < 0$. Then $x_2 + y_2 = 1$. Then the DC deviation by B $(q_1, q_2) = (p_1 - u - \varepsilon, p_2 + u - \varepsilon)$ guarantees that every consumer joins B and yields a profit of $p_1 + p_2 - 2\varepsilon$. Also the DC deviation by A $(p_1, p_2) = (q_1 - u - \varepsilon, q_2 + u - \varepsilon)$ guarantees that every consumer joins A and yields a profit of $q_1 + q_2 - 2\varepsilon$. Note that $(p_1 + p_2 + q_1 + q_2)/2 > p_1x_1 + q_1y_1 + p_2x_2 + q_2y_2$ since $x_1 < y_1$ and $y_2 < x_2$. This implies that $(p_1 + p_2 + q_1 + q_2)/2 > 0$ since profits are nonnegative. Then $p_1 + p_2 + q_1 + q_2 > (p_1 + p_2 + q_1 + q_2)/2$. Then for small enough ε it holds that $p_1 + p_2 - 2\varepsilon + q_1 + q_2 - 2\varepsilon > p_1x_1 + q_1y_1 + p_2x_2 + q_2y_2$. But then either $p_1 + p_2 - 2\varepsilon > q_1y_1 + q_2y_2$ or $q_1 + q_2 - 2\varepsilon > p_1x_1 + p_2x_2$ (or both), making at least one of the DC deviations profitable. Now assume that one of the prices on both markets is negative: $q_1, p_2 < 0$ and $p_1, q_2 > 0$. Then it is analogous to the previous step to show that for small enough ε at least one of the DC deviations $(p_1 - u - \varepsilon, p_2 + u - \varepsilon)$ by B and $(q_1 + u - \varepsilon, q_2 - u - \varepsilon)$ by A is profitable. Finally assume that one firm charges two nonnegative prices, while the other charges one negative and one positive price. Wlog let $p_1, p_2, q_2 \geqslant 0$ and $q_1 < 0$. Assume that $p_1 \geqslant p_2$ (the reverse case is analogous). Then $p_1 + (q_1 + q_2)/2 > (p_1 + p_2 + q_1 + q_2)/2$. Also $q_1 + q_2 > (q_1 + q_2)/2$ since $(q_1 + q_2)/2 > q_1y_1 + q_2y_2 \geqslant 0$. Furthermore, $(p_1 + p_2 + q_1 + q_2)/2 > p_1x_1 + q_1y_1 + p_2x_2 + q_2y_2$ since $x_1, y_2 < 1/2$. These establish that $p_1 + q_1 + q_2 > p_1x_1 + q_1y_1 + p_2x_2 + q_2y_2$. Now consider the DC deviation by B $(q_1, q_2) = (p_1 - u - \varepsilon, u - \varepsilon)$. It guarantees that every consumer joins B and yields a profit of $p_1 - 2\varepsilon$. Similarly, the DC deviation by A $(p_1, p_2) = (q_1 + u - \varepsilon, q_2 - u - \varepsilon)$ guarantees that every consumer joins A and yields a profit of $q_1 + q_2 - 2\varepsilon$. But for small enough ε it holds that $p_1 - 2\varepsilon + q_1 + q_2 - 2\varepsilon > p_1x_1 + q_1y_1 + p_2x_2 + q_2y_2$ and therefore at least one of the two DC deviations above has to be profitable. QED.

Lemma 1 *If $p << q$ then in any coalitionally rationalizable outcome of the subgame with prices (p_1, p_2, q_1, q_2) it has to be that $y_1, y_2 = 0$ (similarly if $p >> q$ then $x_1, x_2 = 0$). If $p_1, p_2 \leqslant u$, neither $p << q$ nor $p >> q$ and $p_1 - u \leqslant q_1$ and $p_2 - u \leqslant q_2$ then $x_1 = x_2 = 1$ is a coalitionally rationalizable Nash equilibrium of the consumer subgame with prices (p_1, p_2, q_1, q_2). Similarly if $q_1, q_2 \leqslant u$, neither $p << q$ nor $p >> q$ and $q_1 - u \leqslant p_1$ and $q_2 - u \leqslant p_2$ then $y_1 = y_2 = 1$ is a coalitionally rationalizable Nash equilibrium of the consumer subgame with prices (p_1, p_2, q_1, q_2).*

Proof.

If $q_1, q_2 \leq u$ then $p_1, p_2 < u$ and therefore joining A is a supported restriction for the coalition of all consumers. If $q_k > u$ for some $k = 1, 2$ then not joining B is a supported restriction for every side k consumer. But then joining B can never be a best response for side $-k$ consumers, which establishes that joining B is not rationalizable for any consumer and therefore not coalitionally rationalizable.

For the second claim, note that the proposed profile is an equilibrium, because a consumer on side 1 gets utility $u - p_1$ which is by assumption larger than or equal to 0, what he would get if not joining any network, and $-q_1$ which he would get if joining firm B. Similarly for consumers on side 2. Since neither $p << q$ nor $p >> q$, for some $k \in \{0, 1\}$ $p_k \leq q_k$. Wlog assume $p_1 \leq q_1$. Let $C = \times_{i \in [0,1]} C_1^i \times_{i \in [0,1]} C_2^i$ be a product subset of the set of strategies in the subgame and $\{A\} \in C_k^i \ \forall\, i \in [0, 1]$ and $k = 1, 2$. Then the maximum utility that is possible for a side 1 consumer in the subgame is attained if he chooses A and all consumers on side 2 choose A. Therefore there is no supported restriction $D = \times_{i \in [0,1]} D_1^i \times_{i \in [0,1]} D_2^i$ given C such that $\{A\} \notin D_1^i$ for some $i \in [0, 1]$. But then conditions $p_2 - u \leq q_2$ and $p_2 - u \leq q_2$ also imply that there is no supported restriction $D = \times_{i \in [0,1]} D_1^i \times_{i \in [0,1]} D_2^i$ given C such that $\{A\} \notin D_2^i$ for some $i \in [0, 1]$. An iterative argument then establishes that $\{A\} \in A_k^* \ \forall\, i \in [0, 1]$ and $k = 1, 2$ where $A^* = \times_{i \in [0,1]} A_1^* \times_{i \in [0,1]} A_2^*$ is the set of coalitionally rationalizable strategies of the subgame. This concludes that the above profile is both equilibrium of the subgame and coalitionally rationalizable. QED

Lemma 2 *Market shutdown cannot be in CRSE.*

Proof.

Theorem 6 establishes that if there is market shutdown in a subgame perfect Nash equilibrium, then $p_1 = p_2 = q_1 = q_2 = 0$. But if $p_1 = p_2 = q_1 = q_2 = 0$, then joining a network is a supported restriction by the coalition of all consumers (it gives an expected payoff of at least $u/2$, while staying out gives a payoff of 0). QED

Lemma 3 *If in some crspe no consumer joins network B then $p_1 = -p_2$, $p_1 \in [-u, u]$ and $x_1 = x_2 = 1$. Similarly if in some crspe no consumer joins network A then $q_1 = -q_2$, $q_1 \in [-u, u]$ and $y_1 = y_2 = 1$.*

Proof.

Assume no consumer joins B. The other case is symmetric.

Suppose first that $p_1 \neq -p_2$. If both $p_1 > 0$ and $p_2 > 0$ then the former assumption implies that $p_1 + p_2 > 0$ and therefore the deviation $(\min(u, p_1) - \varepsilon, \min(u, p_2) - \varepsilon)$ by B is profitable for small enough $\varepsilon > 0$, since by lemma [attila1] it attracts all consumers. Assume now that $p_k < 0$ for $k = 1, 2$. Then $x_k = 1$ since joining A is always a better response than not joining any network. Then since A cannot have negative profit in equilibrium, it has to be that $p_{-k} > -p_k$. But then the deviation $p_1 - \varepsilon, p_2 - \varepsilon$ by B is profitable for small enough $\varepsilon > 0$, since by lemma [attila1] it attracts all consumers. Therefore $p_1 \neq -p_2$ cannot be in a crspe in which no consumer joins B.

Suppose now that $p_1 > u$. Since $p_2 = -p_1$, $p_2 < 0$. Then all side 2 consumers join A in this equilibrium since joining A is always better than not joining any network, but no side 1 consumers, since no joining any network is always strictly better for them. But then A gets negative profit in this equilibrium, a contradiction. A similar argument shows that it cannot be that $p_1 < -u$.

Assume now that it is not the case that $x_1 = x_2 = 1$. If $p_1 < 0$ then it has to be that $x_1 = 1$. Then $x_2 < 1$ and $p_1 = -p_2$ imply that firm A gets negative profit in equilibrium, a contradiction. A similar argument shows that it cannot be that $p_2 < 0$. Finally if $p_1 = 0$ then since no consumer joins B and by lemma 2 market shutdown cannot be in crspe, it has to be that every consumer joins A (otherwise some consumers are not in optimum). QED

Lemma 4 *If in some crspe both networks have a positive market share then*

$$p_1 = q_1 \in [-u/2, u/2],\ p_2 = q_2 = -p_1 \text{ and } x_1 = y_1 = x_2 = y_2 = 1/2.$$

Proof:

By theorem 6 in every subgame perfect Nash equilibrium with both networks having positive market shares it has to be that $p_k = q_k$ for $k = 1, 2$ and $x_1 = y_1 = x_2 = y_2 = 1/2$. Also by theorem 6 either $p_1 = q_1 \in [-u/2, u/2]$ and $p_2 = q_2 = -p_1$ or $p_1 = p_2 = q_1 = q_2 = \hat{p} \in 0, \frac{u}{2}$. But if $\hat{p} > 0$ then deviating to $(\hat{p} - \varepsilon, \hat{p} - \varepsilon)$ is profitable to either firm for small enough $\varepsilon > 0$ since by lemma [attila1] this deviation attracts all consumers and therefore the deviators profit is $2(\hat{p} - \varepsilon)$ which for small enough $\varepsilon > 0$ is larger than $\hat{p}$, the profit that firms get in the proposed equilibrium. QED

Proof of Theorem 7:

Lemmas 2,3,4 establish the first part of the claim.

Consider now a profile such that $p_1 = -p_2$, $p_1 \in [-u, u]$, $q_1 = q_2 = 0$ and consumers' strategies are the following:

-all consumers join A if prices are (p_1, p_2, q_1', q_2') such that $q_1' > p_1 - u$, $q_2' > p_2 - u$ and it is not the case that both $q_1' < p_1$ and $q_2' < p_2$

-all consumers join B if prices are $(p_1', p_2', 0, 0)$ such that $p_1', p_2' > -u$ and it is not the case that p_1' and p_2' are both negative

-consumers play some arbitrary coalitionally rationalizable Nash equilibrium in every other consumer subgame.

By lemma 1 the consumers play a coalitionally rationalizable Nash equilibrium in every subgame in the above profile. Firm A can only get positive market share after a deviation if either it sets two negative prices, or if it sets a price less than $-u$. In either case he gets negative profit, so the deviation is unprofitable. If B deviates to $(q_1', q_2') << (p_1, p_2)$ it gets all consumers and its profit is strictly negative. If he sets $q_1' < p_1 - u$ and $p_1 < 0$ then since no consumer on side 2 joins him if $q_2' > u$, he ends up with negative profit. If he sets $q_1' < p_1 - u$ and $p_1 > 0$ then since no consumer on side 2 joins him if $q_2' > p_2 + u$, he ends up with negative profit. A similar argument establishes that setting $q_2' < p_2 - u$ yields a negative profit. After every other deviation B gets a zero market share, which concludes that neither firm has a profitable deviation in this profile and therefore it is a crspe.

A similar argument establishes that for every (q_1, q_2) such that $q_1 = -q_2$, $q_1 \in [-u, u]$ there is a crspe such that firm B sets prices (q_1, q_2) and all consumers join B.

Consider now a profile such that $p_k = q_k = -p_{-k} = -q_{-k} \in [-u/2, u/2]$ for $k = 1, 2$ and consumers follow the following strategies:

-after prices (p_1, p_2, q_1, q_2) the mass of people joining both networks is $1/2$ on both sides of the market

-all consumers join A if prices are (p_1, p_2, q_1', q_2') such that $q_1' > p_1 - u$, $q_2' > p_2 - u$ and it is not the case that both $q_1' < p_1$ and $q_2' < p_2$

-all consumers join B if prices are (p_1', p_2', q_1, q_2) such that $p_1' > q_1 - u$, $p_2' > q_2 - u$ and it is not the case that both $p_1' < q_1$ and $p_2' < q_2$

-consumers play some arbitrary coalitionally rationalizable Nash equilibrium in every other consumer subgame.

Consider the subgame after price announcements (p_1, p_2, q_1, q_2). Then both the profile in which every consumer joins A and the one in which every consumer joins B yield the best possible payoff to every consumer in this subgame and therefore both joining A and joining B are coalitionally rationalizable in the subgame for every consumer. Furthermore, $1/2$ of the consumers joining A and $1/2$ of the consumers joining B constitutes a

Nash equilibrium in the subgame and therefore the above profile specifies a coalitionally rational Nash equilibrium in this subgame. Lemma 1 then establishes that consumers play a coalitionally rationalizable Nash equilibrium in every other subgame.

If B deviates to $q_1' < p_1$ and $q_2' < p_2$ then it attracts all consumers and gets a strictly negative profit. If he sets a price $q_1' < q_1 - u$ and $q_1 < 0$ then since no consumer on side 2 joins him if $q_2' > u$, he ends up with negative profit. If he sets $q_1' < p_1 - u$ and $p_1 > 0$ then since no consumer on side 2 joins him if $q_2' > p_2 + u$, he ends up with negative profit. Any other deviation results in B getting a 0 market share. This concludes that there is no profitable deviation for B. A symmetric argument establishes that there is no profitable deviation for A and therefore the above profile is a crspe. QED

Proof of Theorem 8:

If the monopolist charges prices that are lower than l on both sides of the market, then joining the network is a supported restriction for the coalition of all consumers. The supremum of the profit the firm in this price range is $2l$ and the firm can get a profit arbitrarily close to it by charging prices $(l-\varepsilon, l-\varepsilon)$ for small enough $\varepsilon > 0$. If the firm sets both prices to be smaller than ah then joining the network is a supported restriction for the coalition of consumers that involve the high types from both sides of the market. Therefore the monopolist can guarantee a profit arbitrarily close to $2a^2h$ by charging prices $(ah-\varepsilon, ah-\varepsilon)$ for small enough $\varepsilon > 0$. If the firm charges a price higher than h on one side of the market, then it is never a best response for any consumer on that side to join the network. Therefore consumers on the other side do not join the network if they are charged a positive price. Therefore the maximum profit the monopolist can get if charging a price higher than h is 0, which is strictly smaller than $2a^2h$. Therefore in a crspe the monopolist never charges a price higher than h.

Consider first the case that $ah > l$. If the monopolist charges a price higher than ah on one side, then only high types can join the network in equilibrium. Furthermore, consumers on this side only join in equilibrium if at least some low types join the network from the other side. For that to be possible in equilibrium, the price on the other side cannot exceed al. The above imply that if the firm charges a price higher than ah on one side, then its profit cannot exceed $ah + al$ (charging more than h on that side or more than al on the other side would imply that no consumers join the network). But note that the firm can get a profit arbitrarily close to this amount by charging $(h-\varepsilon, al-\varepsilon)$ for small enough $\varepsilon > 0$, since then joining the network is a supported restriction for the coalition of consumers involving all high types on side 1 and all consumers on side 2.

Consider now the case that $l > ah$. If the monopolist charges a price higher than l on one side, the same arguments as above establish that its profit cannot exceed $ah + al$, but it can get a profit arbitrarily close to it by charging $(h - \varepsilon, al - \varepsilon)$ for small enough $\varepsilon > 0$.

This concludes that the monopolist can never get a higher profit in a crspe then $\max(2l, 2a^2h, ah + al)$, but if consumers are coalitionally rational, then it can always guarantee a profit arbitrarily close to $\max(2l, 2a^2h, ah + al)$. But then its profit in every crspe has to be equal to $\max(2l, 2a^2h, ah + al)$. This establishes that the prices charged by the monopolist are either (l, l) or (ah, ah) or (h, al) or (al, h) in any crspe, and in the first case all consumers join the network, in the second all high types join, in the third all high types on side 1 and all consumers on side 2 join, while in the fourth all high types on side 2 and all consumers on side 1 join. Note that every consumer who joins the network in the proposed profiles in the above consumer subgames gets nonnegative utility, while not joining the network never gives utility larger than 0. Also for those consumers who do not join the network in the above profiles, joining the network is not rationalizable. Therefore these strategy profiles in the corresponding consumer subgames are coalitionally rationalizable Nash equilibria. Then if $2l = \max(2l, 2a^2h, ah+al)$ then there is a crspe in which the firm charges (l, l), otherwise there is not. Similarly if $2a^2h = \max(2l, 2a^2h, ah+al)$ then there is a crspe in which the firm charges (ah, ah), otherwise there is not, and if $ah + al = \max(2l, 2a^2h, ah + al)$ then there is a crspe in which the firm charges (h, al) or (al, h), otherwise there is not.

Suppose $2a^2h > al + ah$. Since $a > 0$, this is equivalent to $(2a - 1)h > l$. The latter implies $a^2h > l$ since $a^2h - (2a - 1)h = (a - 1)^2h > 0$. Therefore $(2a - 1)h > l$ implies that there is a crspe in which prices are (ah, ah) and if $(2a - 1)h > l$ then in every crspe prices are (ah, ah). If $(2a - 1)h < l$ then prices cannot be (ah, ah) in crspe.

Suppose now that $ah + al$ 6 $2l$. It is equivalent to $l > \frac{a}{2-a}h$. The latter implies $l > a^2h$ since $\frac{a}{2-a}h - a^2h = ah\frac{1-2a+a^2}{2-a} > 0$. Therefore $2a^2h < 2l$. This establishes that if $l > \frac{a}{2-a}h$ then there is a crspe in which prices are (l, l), if $l > \frac{a}{2-a}h$ then in every crspe prices are (h, h) and if $l > \frac{a}{2-a}h$ then there is no crspe in which prices are (l, l).

Note that $(2a - 1)h < \frac{a}{2-a}h$.

The above also establish that if $l \in [(2a-1)h, \frac{a}{2-a}h]$ then there is a crspe in which one price is h and the other is al, if $l \in ((2a-1)h, \frac{a}{2-a}h)$ then all crspe are like that and if $l \notin [(2a-1)h, \frac{a}{2-a}h]$ then there is no crspe like that. QED.

Proof of Theorem 9:

Suppose the monopolist establishes two networks, A and B, charging prices (p_1, p_2) and (q_1, q_2) respectively and denote with (x_1, x_2) A's market shares and with (y_1, y_2) B's market shares. If $(p_1, p_2) << (q_1, q_2)$ or viceversa, one of the two networks would be empty by coalitional rationality[8] and therefore he could not get higher total profits than what he gets in the case where he has only one network. If he charges $(p_1, p_2) = (q_1, q_2)$, it is compatible with coalitional rationality that either consumers stay out of the market (if prices are nonnegative), in which case profits are null, or that one network is empty, in which case the monopolist cannot get more profits than if he had only one network, or that consumers equally split between the two networks. Even in this last case, the monopolist could do better by offering just one network: he could get the same number of customers and increase prices because the gross surplus that consumers would receive by converging on one network would be higher. Finally, the monopolist might charge prices (p_1, p_2), (q_1, q_2) such that $p_1 \geq q_1$ and $p_2 \leq q_2$. First, notice that it has to be the case that $x_1 \leq y_1$ and $y_2 \leq x_2$, otherwise one network would be empty and the monopolist might as well establish just one network. Let $a \leq \frac{1}{2}$ and consider the following strategy: the monopolist charges prices such that network A attracts high types of side 1 and low types of side 2 and network B the remaining consumers, with profits

$$\pi = p_1 a + p_2 (1-a) + q_1 (1-a) + q_2 a \tag{17}$$

Such a strategy must satisfy incentive compatibility and individual rational-

[8] It is a supported restriction for the coalition of all consumers of both sides to restrict their choices to either join the cheaper network or none.

ity constraints for both high and low types:

$$U_h^1(A) = h(1-a) - p_1 \geq U_h^1(B) = ha - q_1 \tag{18}$$
$$U_h^2(B) = h(1-a) - q_2 \geq U_h^2(A) = ha - p_2 \tag{19}$$
$$U_h^1(A) = h(1-a) - p_1 \geq 0 \tag{20}$$
$$U_h^2(B) = h(1-a) - q_2 \geq 0 \tag{21}$$
$$U_l^1(A) = l(1-a) - p_1 \leq U_l^1(B) = la - q_1 \tag{22}$$
$$U_l^2(B) = l(1-a) - q_2 \leq U_l^2(A) = la - p_2 \tag{23}$$
$$U_l^1(B) = la - q_1 \geq 0 \tag{24}$$
$$U_l^2(A) = la - p_2 \geq 0 \tag{25}$$

Maximizing profits subject to this set of constraints we first observe that the symmetry of the problem guarantees a symmetric solution where

$$p_1 = q_2 \tag{26}$$
$$p_2 = q_1 \tag{27}$$

The solution of this linear programming problem is

$$p_1 = q_2 = h(1-2a) + al \tag{28}$$
$$p_2 = q_1 = la \tag{29}$$

with the IR of the low types and the IC of the high types are binding. Associated profits are

$$\pi = 2\left[ah(1-2a) + al\right] \tag{30}$$

First, notice that

$$2\left[ah(1-2a) + al\right] > al + ah \Rightarrow l > (4a-1)h \tag{31}$$

therefore for values of l smaller than $(4a-1)h$ the monopolist would rather establish only one network and charge (h, al) or (al, h) than establish two networks and charge $p_1 = q_2 = h(1-2a) + al$, $p_2 = q_1 = al$. Moreover, this also establishes that for $l \in (4a-1)h, \frac{a}{2-a}h$ establishing two networks and charging $p_1 = q_2 = h(1-2a) + al$, $p_2 = q_1 = al$ is more profitable than any strategy involving the presence of just one network. For $l \geq \frac{a}{2-a}h$, we know that the most profitable strategy for a monopolist who operates only one network is to charge (l, l) and attract all consumers on both sides with profits $\pi = 2l$. Since

$$2\left[ah(1-2a) + al\right] > 2l \Rightarrow l < \frac{a(1-2a)}{1-a}h \tag{32}$$

we can conclude that for $l \in \left(\frac{a}{2-a}h, \frac{a(1-2a)}{1-a}h\right)$ establishing two networks and charging $p_1 = q_2 = h(1-2a) + al$, $p_2 = q_1 = al$ is more profitable than any strategy involving the presence of just one network. Notice that such a range exists iff $a \in \left(0, 1 - \frac{\sqrt{2}}{2}\right)$.

Now that we have shown that for some values of the parameters this strategy is more profitable than any strategy involving the presence of only one network, it remains to show that the monopolist has no profitable deviation involving the presence of two networks. We have already shown that no strategy of the type $(p_1, p_2) << (q_1, q_2)$ or $(p_1, p_2) = (q_1, q_2)$ can improve on the strategies involving only one network, and therefore they cannot improve on our candidate equilibrium either. Among the strategies of the type $p_1 \geq q_1$ and $p_2 \leq q_2$, we already know that only the ones which induce $x_1 \leq y_1$ and $y_2 \leq x_2$ are potentially profitable. Among these, those who satisfy the set of constraints listed above are all weakly dominated by our candidate equilibrium by construction. Let us consider those who violate the IR of some group. Either the IR is violated for both high and low types, in which case the monopolist gets zero profits, or only the one of the low types is. In this case, on at least one side only high types are willing to enter the market. Either they converge on one network, and therefore the monopolist is not improving with respect to the situation where he has just one network, or they split among the two networks. Even in this case, the monopolist would be better off offering just one network because these consumers would enjoy a higher gross surplus and would be willing to pay a higher price. We have to check that the remaining strategies which induce full coverage of the population and $0 < x_1 \leq y_1$ and $0 < y_2 \leq x_2$ do not constitute profitable deviations. Given the assumption that $a < \frac{1}{2}$, conditions $p_1 \geq q_1, p_2 \leq q_2, 0 < x_1 \leq y_1$ and $0 < y_2 \leq x_2$ can be satisfied if network A attracts all high types consumers and some low types of side 1 and part of the low types of side 2 and network B the remaining consumers, with profits

$$\pi = p_1 x_1 + p_2 x_2 + q_1 y_1 + q_2 y_2 \tag{33}$$

By the symmetry of the problem, the solution will be such that

$$\begin{aligned} p_1 &= q_2 && (34) \\ p_2 &= q_1 && (35) \\ x_1 &= y_2 \leq \frac{1}{2} && (36) \\ x_2 &= y_1 = 1 - y_2 && (37) \end{aligned}$$

and the set of constraints that need to be satisfied is

$$U_{\mathsf{h}}^{1}(A) = h(1-y_2) - p_1 \geq U_{\mathsf{h}}^{1}(B) = hy_2 - q_1 \quad (38)$$
$$U_{\mathsf{l}}^{1}(A) = l(1-y_2) - p_1 = U_{\mathsf{l}}^{1}(B) = ly_2 - q_1 \quad (39)$$
$$U_{\mathsf{h}}^{1}(A) = h(1-y_2) - p_1 \geq 0 \quad (40)$$
$$U_{\mathsf{l}}^{1}(A) = l(1-y_2) - p_1 = U_{\mathsf{l}}^{1}(B) = ly_2 - q_1 \geq 0 \quad (41)$$

The solution to this constrained maximization problem is

$$p_1 = q_2 = \frac{l}{2} \quad (42)$$
$$p_2 = q_1 = \frac{l}{2} \quad (43)$$
$$x_1 = y_2 = \frac{1}{2} \quad (44)$$
$$x_2 = y_1 = \frac{1}{2} \quad (45)$$
$$\pi = l \quad (46)$$

therefore this deviation is not profitable, since we are considering the case where our candidate equilibrium strategy gives profits that are higher than those attainable with only one network, and in particular higher than $2l$.

Another situation where conditions $p_1 \geq q_1$,$p_2 \leq q_2, 0 < x_1 \leq y_1$ and $0 < y_2 \leq x_2$ are satisfied and there is full coverage is if network A attracts some high types of side 1, part of the high types of type 2 and all the low types of side 2 and network B the remaining consumers, with profits

$$\pi = p_1 x_1 + p_2 x_2 + q_1 y_1 + q_2 y_2 \quad (47)$$

By the symmetry of the problem, the solution will be such that

$$p_1 = q_2 \quad (48)$$
$$p_2 = q_1 \quad (49)$$
$$x_1 = y_2 \leq \frac{1}{2} \quad (50)$$
$$x_2 = y_1 = 1 - y_2 \quad (51)$$

and the set of constraints that need to be satisfied is

$$U_{\mathsf{h}}^{1}(A) = h(1-y_2) - p_1 = U_{\mathsf{h}}^{1}(B) = hy_2 - q_1 \quad (52)$$
$$U_{\mathsf{l}}^{1}(A) = l(1-y_2) - p_1 \leq U_{\mathsf{l}}^{1}(B) = ly_2 - q_1 \quad (53)$$
$$U_{\mathsf{h}}^{1}(A) = h(1-y_2) - p_1 = U_{\mathsf{h}}^{1}(B) = hy_2 - q_1 \geq 0 \quad (54)$$
$$U_{\mathsf{l}}^{1}(B) = ly_2 - q_1 \geq 0 \quad (55)$$

The solution is

$$
\begin{aligned}
p_1 &= q_2 = \frac{h(3h-l)-(h+l)(h-l)}{4h} && (56)\\
p_2 &= q_1 = \frac{l(h+l)}{4h} && (57)\\
x_1 &= y_2 = \frac{h+l}{4h} && (58)\\
x_2 &= y_1 = \frac{3h-l}{4h} && (59)\\
\pi &= \frac{(h+l)^2}{4h} && (60)
\end{aligned}
$$

and it exists only if $x_1 = y_2 = \frac{h+l}{4h} < a$ since we assumed that network A attracts only part of the high types of side 1. But

$$\frac{h+l}{4h} < a \Rightarrow \frac{(h+l)^2}{4h} < a(h+l) \tag{61}$$

The last term of the inequality is the profit the monopolist might achieve by establishing just one network and attracting all consumers of one side and only high types of the other side. Since we are considering the case where our candidate equilibrium is more profitable than any strategy involving only one network, we can conclude that this deviation is not profitable. The last case where conditions $p_1 \geq q_1$,$p_2 \leq q_2, 0 < x_1 \leq y_1$ and $0 < y_2 \leq x_2$ are satisfied and there is full coverage is if network A attracts some low types of side 1 (a fraction smaller than $\frac{1}{2}$ of the total population), all the high types of type 2 and some low types of side 2 and network B the remaining consumers, with profits

$$\pi = p_1 x_1 + p_2 x_2 + q_1 y_1 + q_2 y_2 \tag{62}$$

By the symmetry of the problem, the solution will be such that

$$
\begin{aligned}
p_1 &= q_2 && (63)\\
p_2 &= q_1 && (64)\\
x_1 &= y_2 \leq \frac{1}{2} && (65)\\
x_2 &= y_1 = 1 - y_2 && (66)
\end{aligned}
$$

and the set of constraints that need to be satisfied is

$$\begin{aligned} U_{\mathsf{h}}^{1}(A) &= h(1-y_2)-p_1 \leq U_{\mathsf{h}}^{1}(B) = hy_2-q_1 && (67)\\ U_{\mathsf{l}}^{1}(A) &= l(1-y_2)-p_1 = U_{\mathsf{l}}^{1}(B) = ly_2-q_1 && (68)\\ U_{\mathsf{h}}^{1}(B) &= hy_2-q_1 \geq 0 && (69)\\ U_{\mathsf{l}}^{1}(A) &= l(1-y_2)-p_1 = U_{\mathsf{l}}^{1}(B) = ly_2-q_1 \geq 0 && (70) \end{aligned}$$

This set of constraints has no solution because the IC of the low types implies

$$p_1 - q_1 = l(1-2y_2) \tag{71}$$

which is not compatible with the IC of the high types, which in turns can be rewritten as

$$p_1 - q_1 \geq h(1-2y_2) > l(1-2y_2) \tag{72}$$

Finally, we have shown that there is no profitable deviation from our candidate equilibrium. QED

References

[1] AMBRUS, A. (2002): "Coalitional rationality," *mimeo Princeton University*

[2] AMBRUS, A. (2003): "Dynamic coalitional agreements in multi-stage games," *mimeo Princeton University and Yale University*

[3] BAUMOL, W.J., J. PANZAR and R. WILLIG (1982): "*Contestable Markets and the Theory of Industry Structure",* Harcourt Brace, Jovanovich, New York, NY.

[4] BENTAL, B and M. SPIEGEL (1995): "Network Competition, Product Quality, and Market Coverage in the Presence of Network Externalities," *Journal of Industrial Economics,* 43, 197-208.

[5] BERNHEIM, B.D., B. PELEG and M.D.WHINSTON (1987): "Coalition-Proof Nash Equilibria I. Concepts," *Journal of Economic Theory,* 42, 1-12.

[6] CAILLAUD, B. and B. JULLIEN (2001a): "Chicken & Egg: Competing Matchmakers," *mimeo IDEI Tolouse.*

[7] CAILLAUD, B. and B. JULLIEN (2001b): "Competing Cybermediaries," *European Economic Review,* 45, 797-808.

[8] DE PALMA, A. and L. LERUTH (1993): "Equilibrium in Competing Networks with Differentiated Products," *Transportation Science,* 27, 73-80.

[9] ECONOMIDES, N. (1996): "The Economics of Networks," *International Journal of Industrial Organization*, 14, 673-699

[10] ELLISON, G., D. FUDENBERG and M. MOBIUS (2002): "Competing Auctions," *mimeo MIT and Harvard University.*

[11] FARRELL, J. and P. KLEMPERER (2001): "Coordination and Lock-In: Competition with Switching Costs and Network Effects," *mimeo University of California at Berkeley and Nuffield College, Oxford University.*

[12] FARRELL, J. and G. SALONER (1985): "Standardization, Compatibility and Innovation," *Rand Journal of Economics,* 16, 70-83.

[13] JULLIEN, B. (2001): "Competing in Network Industries: Divide and Conquer," *mimeo IDEI Tolouse.*

[14] KATZ, M. and C. SHAPIRO (1985): "Network Externalities, Competition and Compatibility," *American Economic Review,* 75, 424-440

[15] KATZ, M. and C. SHAPIRO (1994): "System Competition and Network Effects", *Journal of Economic Perspectives,* 8, 93-115.

[16] LAFFONT, J.J., S. MARCUS, P. REY and J. TIROLE (2001): "Internet Interconnection and the Off-Net-Cost Pricing Principle," *mimeo IDEI Tolouse.*

[17] LEIBENSTEIN, H. (1950): "Bandwagon, Snob and Veblen Effects in the Theory of Consumers' Demand," *Quarterly Journal of Economics,* 64, 183-207.

[18] PARKER, G. and M. VAN ALSTYNE (2000): "Information Complements, Substitutes and Strategic Product Design," *mimeo Tulane University and University of Michigan".*

[19] ROCHET, J.C. and J. TIROLE (2002a): "Cooperation among Competitors: Some Economics of Payment Card Associations," *RAND Journal of Economics,* 33, 549-570.

[20] ROCHET, J.C. and J. TIROLE (2002b): "Platform Competition in Two-Sided Markets," *mimeo IDEI Tolouse.*

[21] ROHLFS, J. (1974): "A Theory of Interdependent Demand for a Communications Service," *Bell Journal of Economics,* 5, 16-37.

[22] SALANIE, B. (1997): *"The economics of contracts: A primer,"* MIT Press, Cambridge, MA.

[23] SCHMALENSEE, R. (2001): "Payment systems and Interchange Fees," *NBER Working Paper 8256.*

[24] VARIAN, H. (1988): "Price Discrimination," in Schmalensee, R. and R. Willig (editors) *Handbook of Industrial Organization,* Elsevier, Amsterdam.